On finitely presented expansions of semigroups, algebras, and groups

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Plan

- Formulation of the problem
- Immune algebras
- The NFP theorem
- Applications to
  1. semigroups,
  2. algebras, and
  3. groups (through Golod-Shafarevich).

Algebras

The word algebra is used in two ways. One is **algebras** are rings that are vector spaces over fields. The other is:

Definition

An **algebra** is an algebraic structure of the form:

$$\mathcal{A} = (A; f_1, \ldots, f_n, c_1, \ldots, c_r), \text{ where } A \neq \emptyset, \text{ and}$$

$f_1, \ldots, f_n$ are operations and $c_1, \ldots, c_r$ are constants on A.

The term algebra

By $\mathcal{F}$ denote the term algebra built from constants.

Properties of $\mathcal{F}$:

1. The term algebra $\mathcal{F}$ is finitely generated and computable.
2. The term algebra $\mathcal{F}$ is **universal** and unique.

Presentations

Definition

An **equational presentation** S is a **finite** set of equations

$$t = p,$$

where t and p are terms (that might contain variables).

Let $E(S)$ be the congruence relation on $\mathcal{F}$ generated by S. Set:

$$\mathcal{F}_S = \mathcal{F}/E(S).$$

Definition

The algebra $\mathcal{F}_S$ is called **finitely presented** by S.

Properties of $\mathcal{F}_S$:

1. The algebra $\mathcal{F}_S$ satisfies S.
2. The equality relation on $\mathcal{F}_S$ is $E(S)$ and $E(S)$ is c.e..
3. The operations of $\mathcal{F}_S$ are computable and respect $E(S)$.
4. The algebra $\mathcal{F}_S$ is universal and unique.

The finite presentability problem (general formulation)

Given an algebra $\mathcal{A}$, is $\mathcal{A}$ finitely presented?

Necessarily, $\mathcal{A}$ must be f.g., must have c.e. equality, and operations must be computable that respect the equality.

Necessary assumptions for the rest of the talk:

1. We view algebras $\mathcal{A}$ as quotients

$$\mathcal{A} = \mathcal{F}/E.$$

2. The atomic operations of $\mathcal{A}$ are computable and respect E.

Examples of non-finitely presented algebras:

- The groups $Z_k \wr Z^n$, with $k, n > 1$, are not finitely presented (Baumslag, 1960).
- The algebra $(\omega; x+1, 2^x, 0)$ is not finitely presnted (Bergstra and Tucker, 1979).

Expansions

Definition

An **expansion** of $\mathcal{A} = (A; f_1, \ldots, f_n, c_1, \ldots, c_r)$ is any algebra of the form $\mathcal{A}' = (A; f_1, \ldots, f_n, g_1, \ldots, g_s, c_1, \ldots, c_r)$.

Does every f.g. and c.e. algebra possess a finitely presented expansion?

Answer: *Yes* for computable algebras.

Theorem (Bergstra-Tucker, $\approx$ 1980)

Any computable algebra has a finitely presented expansion.

Formulation of the problem

The problem was stated in the early 80s
(by Bergstra-Tucker, and independently by Goncharov):

Does every f.g. and c.e. algebra have
a finitely presented expansion?

Answer: *No*.

Theorem (Kassymov (1988), Khoussainov(1994))

There exist f.g. and c.e. algebras that have no finitely presented expansions.

Reformulation of the problem

Can such examples be found in typical algebraic structures such as semigroups, groups, or algebras?

Immune algebras

Definition

An algebra $\mathcal{A} = \mathcal{F}/E$ is **effectively infinite** if there is an infinite c.e. sequence $t_0, t_1, \ldots$ of pairwise distinct elements of $\mathcal{A}$.

Otherwise, call the algebra $\mathcal{A}$ **immune** if the algebra is infinite.

Immune semigroups

Theorem (with D. Hirschfeldt)

There exists a f.g., c.e., and immune semigroup.

Proof (Outline). Consider $\mathbf{A} = (\{a,b\}^\star; \circ)$ the free semigroup. Let $X \subseteq \{a,b\}^\star$ be a nonempty subset. Define:

$$u \equiv_X v \iff u = v \ \vee \ u \text{ and } v \text{ have substrings from } X.$$

The semigroup $\mathbf{A}(X)/\equiv_X$ is well defined.

Lemma

There is a simple set X such that $\mathbf{A}(X)/\equiv_X$ is infinite, and hence immune.

The case of groups

A natural question arises:

Are there f.g. immune groups?

Such groups answer the generalised Burnside problem.

Theorem (Miasnikov, Osin, 2011)

There are f.g., c.e., immune groups.

In group theory these groups are now called **Dehn monsters.**

Separator Lemma

Lemma (Separator Lemma)

If $\mathcal{A} = \mathcal{F}/E$ is **residually finite** *then for all distinct $x, y \in A$ there is a separator subset $S(x, y) \subset F$ such that:*

1. *$S(x, y)$ is computable and E-closed.*
2. *$x \in S(x, y)$ and $y \in F \setminus S(x, y)$.*

Note: Immunity is not used in the proof. Also, there are no computability-theoretic assumptions on E.

The Main Lemma

Lemma (The main lemma)

If $\mathcal{A} = \mathcal{F}/E$ is immune and residually finite, then so are all expansions of $\mathcal{A}$.

Let x, y be distinct elements of an expansion $\mathcal{A}'$ of $\mathcal{A}$.

Define:

$a \equiv_{(x,y)} b \iff$ no elements in $S(x, y)$ and its complement are identified by the congruence relation on $\mathcal{A}'$ generated by (a, b).

Properties of $\equiv_{(x,y)}$

The relation has the following properties:

1. $\equiv_{(x,y)}$ is a congruence relation on $\mathcal{A}'$.
2. $\equiv_{(x,y)}$ is a co-c.e. relation.
3. The quotient $\mathcal{A}' \,/\, \equiv_{(x,y)}$ is finite.
4. In $\mathcal{A}' \,/\, \equiv_{(x,y)}$ the images of x and y are distinct.

The non-finite presentability theorem

Theorem (The NFP Theorem)

Let $\mathcal{A} = \mathcal{F}/E$ be a f.g., c.e., immune, and residually finite algebra. Then $\mathcal{A}$ has no finitely presented expansions.

Proof.

Let $\mathcal{A}'$ be a finitely presented expansion.
Since $\mathcal{A}'$ is residually finite, the equality E in $\mathcal{A}'$ is decidable (by Malcev/McKenzie theorem). $\square$

Application of the NFP Theorem: Semigroups case

Corollary (Semigroups case)

There exists a f.g. c.e. and immune semigroup that has no finitely presented expansion.

Proof.

The semigroup $\mathbf{A}(X)$ that we already built is f.g., c.e., immune, and residually finite. Apply the NFP theorem. □

How about algebras and groups?

Now we want to construct finitely generated, computably enumerable, residually finite, and immune algebras and groups.

We use Golod-Shafarevich Theorem.

Golod-Shafarevich Set-Up

Let K be a finite field. Consider the algebra

$$\mathcal{P} = K\langle x_1, x_2, \ldots, x_s\rangle$$

of polynomials in non-commuting variables.

Represent $\mathcal{P}$ as the direct sum

$$\mathcal{P} = \sum_n \mathcal{P}_n$$

where $\mathcal{P}_n$ is the vector space spanned over s^n monomials of degree n.

Golod-Shafarevich Theorem

Let H be a set of homogeneous polynomials.

Let $I = \langle H \rangle$ be the ideal generated by H.

Theorem (Golod Shafarevich)

Let r_n be the number of polynomials in H of degree n.

Let ϵ be such that $0 < \epsilon < s/2$ and for all n we have:

$$r_n \leq \epsilon^2 \cdot (s - 2\epsilon)^{n-2}.$$

Then the algebra

$$\mathcal{A} = \mathcal{P}/I = \sum_n \mathcal{P}_n/I$$

is infinite dimensional.

Application of the NFP Theorem: Algebras case

Corollary (with A. Miasnikov)

There exists a f.g. c.e. and immune algebra that has no finitely presented expansion.

Proof (Outline)
Use Post's type of construction to build a simple set H of homogeneous polynomials. Also, satisfy the assumption of Golod-Shafarevich theorem. The algebra

$$\mathcal{A} = \mathcal{P}/I$$

is residually finite. Apply the NFP theorem.

Application of the NFP Theorem: Groups case

Corollary (with A. Miasnikov)

There exists a f.g., c.e., and immune group G that has no finitely presented expansion.

For the algebra $\mathcal{A}$ built above (over two variables x and y), the semigroup $G = G(\mathcal{A})$ generated by

$$(1 + x)/I \ \ and \ \ (1 + y)/I$$

is a group. The group G is residually finite. Apply the NFP theorem.

Is immunity important?

Answer: *No*.

Theorem

There exists a f.g., c.e., effectively infinite, and residually finite algebra that has no finitely presented expansion.

Answer: *Yes*.
The transversal of the equality relation built in the algebra is immune. This immunity is used in the proof.

Open question

Are there semigroups, algebras, and groups that have no quasi-equational finite presentations?