Topologizing the degree theory

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Theme: The theory of degrees of unsolvability in topological spaces.

...but, who cares for degrees of points in topological spaces?

The final test of every new theory is its success in answering preexistent questions that the theory was not specifically created to answer.

— David Hilbert, "On The Infinite" (1926).

English translation is by van Heijenoort (1967).

The Jayne-Rogers Theorem (1982)

X: absolute Suslin-$\mathcal{F}$ space, Y separable metrizable.
For a function $f : X \to Y$, the following are equivalent:

1. $f^{-1}[A]$ is F_σ for any F_σ set A.
2. f is closed-piecewise continuous.

Conjecture (by many descriptive set theorists): The Jayne-Rogers Theorem can be generalized to all finite levels of the Borel hierarchy.

Theorem (K. 2015)

X, Y: finite dimensional Polish spaces.
For a function $f : X \to Y$, the following are equivalent for $n < 2m$:

1. $f^{-1} : \utilde{\Sigma}^0_{m+1}(Y) \to \utilde{\Sigma}^0_{n+1}(X)$ is continuous.
2. f is $\utilde{\Pi}^0_n$-piecewise $\utilde{\Sigma}^0_{n-m+1}$-measurable.

Why finite dimensional? Because I used the Shore-Slaman Join Theorem
$\mathcal{D}_T \models (\forall a, b)(\exists c \geq a)\ [((\forall \beta < \alpha)\ b \not\leq a^{(\beta)}) \ \to\ (c^{(\alpha)} \leq b \vee a^{(\alpha)} \leq b \vee c)]$.

Why finite dimensional? Because I used the Shore-Slaman Join Theorem for Turing degrees, and...

(J. Miller 2004) The degree structure of the infinite-dimensional space $[0,1]^\omega$ is different from the Turing degrees! (i.e., continuous degrees)

Theorem (Gregoriades-K.-Ng)

X: analytic subset of a Polish space, Y: separable metrizable.
$f^{-1}\utilde{\Sigma}^0_{m+1} \subseteq \utilde{\Sigma}^0_{n+1} \implies f$ is $\utilde{\Pi}^0_{n+1}$-piecewise $\utilde{\Sigma}^0_{n-m+1}$-measurable.

The proof utilizes many results from computability theory and effective descriptive set theory.

- The Shore-Slaman Join Theorem for continuous degrees.
- Almost-totality of continuos degrees.
- The Friedberg Jump Inversion Theorem.
- The Louveau Separation Theorem for $\utilde{\Sigma}^1_1$ sets.

- Some has claimed that a non-flat generalizion of the Jayne-Rogers Theorem is a wrong way to go. (I don't agree, though.)
- B/c Jayne's original motivation came from Banach space theory.
- But even if the claim is true, generalized degree theory is useful.

Theorem (K.-Pauly)

There is a uncountable collection $(X_\lambda)_{\lambda\in\Lambda}$ of compact metric spaces s.t. $\mathcal{B}_n(X_\alpha)$ is not linearly isometric to $\mathcal{B}_n(X_\beta)$ whenever $\alpha \neq \beta$ and $n \in \omega$.

Here, $\mathcal{B}_n(X)$ is the Banach space of Baire class n functions on X.

The proof utilizes the following results:

- A flat generalization of the Jayne-Roges Theorem of finite Borel rank.
- (J. Miller 2004) Every countable Scott set is realized as a lower Turing cone of a non-total continuous degree.

Summary for the generalization of the Turing degree theory:

We thus conclude that J. Miller's theory of continuous degrees (Turing degree theory for metric spaces) passed Hilbert's final test.
That is, the theory succeeded in answering various questions that the theory was not specifically created to answer!

- V. Gregoriades, T. Kihara, and K. M. Ng, "*Turing degrees in Polish spaces and decomposability of Borel functions*", submitted.
- T. Kihara, and A. Pauly, "*Point degree spectra of represented spaces*", submitted.

K.-Pauly generalized the Turing degree theory to general represented spaces (including non-metrizable spaces). In this context, we have already found a large number of applications of non-metrizable topology in the study of enumeration degrees.

- T. Kihara, S. Lempp, K. M. Ng, and A. Pauly, "*Enumeration degrees and non-metrizable topology*", preprint, 93 pages.

Okay, then, what about other notions of degrees, such as ***tt***-degrees?

$y^Y \leq_T x^X \iff$ There is a partial computable function Φ which maps every name of x to a name of y.
$\iff$ There is a partial computable function $f :\subseteq X \to Y$ s.t. $f(x) = y$.

Definition (McNicholl-Rute)

Let X, Y be computable metric spaces. $x \in X$, $y \in Y$.
We write that $y^Y \leq_{tt} x^X$ if there is a total computable function Φ which maps every Cauchy-name of x to a Cauchy-name of y.

Φ does not necessarily induce a total function $f : X \to Y$.
Instead...

Proposition (McNicholl-Rute)

$y^Y \leq_{tt} x^X$ iff there is a partial computable function $f :\subseteq X \to Y$ with a Π^0_1 domain such that $f(x) = y$.

Equivalent definitions of $x \leq_{tt} y$ (McNicholl-Rute)

- There is a total computable function Φ which maps every Cauchy-name of x to a Cauchy-name of y.
- There is a partial computable function $f :\subseteq X \to Y$ with a Π^0_1 domain such that $f(x) = y$.

- Why f cannot be total? The reason comes from topological facts:
- A space Y is an absolute extensor of X if for any closed $P \subseteq X$, any continuous $f : P \to Y$ has a continuous extension $g : X \to Y$.
- $\mathbf{dim}(X) \leq n \iff$ the n-sphere $\mathbb{S}^n$ is an absolute extensor of X.
- $\mathbf{c\text{-}dim}_G(X) \leq n \iff K(G, n)$ is an absolute extensor of X, where $K(G, n)$ is an Eilenberg-Mac Lane complex.
- Indeed, the modern topological dimension theory (such as Dranishnikov's theory of extension dimension) often uses an absolute extensor as a definition of topological dimension.

We claim that the tt-reducibility is one-level below the T-reducibility.

Two spaces X and Y are n-th level Borel isomorphic if

$$\exists \text{ a bijection } f : X \to Y \text{ s.t. } A \in \underset{\sim}{\Sigma}^0_{n+1} \iff f[A] \in \underset{\sim}{\Sigma}^0_{n+1}.$$

As a consequence of the generalized Jayne-Rogers theorem, we get:

Theorem (K.-Pauly)

The following are equivalent for Polish spaces X and Y:

1. $\omega \times X$ and $\omega \times Y$ are second-level Borel isomorphic.
2. X and Y have the same set of T-degrees up to some oracle.

As a consequence of the original Jayne-Rogers theorem, we get:

Theorem (K.)

The following are equivalent for Polish spaces X and Y:

1. $\omega \times X$ and $\omega \times Y$ are first-level Borel isomorphic.
2. X and Y have the same set of tt-degrees up to some oracle.

Thanks to the Jayne-Rogers Theorem, the notion of first level Borel function has found to be very robust:

X, Y: Polish spaces. TFAE for $f : X \to Y$.

- f is a first-level Borel function.
- f is $\utilde{\Delta}^0_2$-measurable.
- f is $\utilde{\Pi}^0_1$-piecewise continuous.
- f is continuous with finite mindchanges.
- f is the discrete limit of continuous functions. (Császár et al. 1978)
- f is a Baire-one-star function. (O'Malley 1977)

Jayne-Rogers (1979) showed that the topological dimension is first-level Borel invariant. As a consequence,

Corollary

If $\mathbf{dim}(X) \neq \mathbf{dim}(Y)$ then $\mathcal{D}_{tt}(X) \neq \mathcal{D}_{tt}(Y)$.

Hereafter we only consider computable (σ-)compact metric spaces.

A point $x \in X$ is n-dimensional if it is contained in an n-dimensional Π^0_1 set.

Proposition

x is n-dimensional and $y \equiv_{tt} x$ then y is also n-dimensional.

Idea of Proof

Effectivize the topological fact that there exists a universal n-dimensional compactum.

A point x is n-Euclidean if $x \equiv_{tt} y$ for some $y \in \mathbb{R}^n$.

Example

- Every n-dimensional point is $(2n + 1)$-Euclidean.
- There is an n-dimensional point which is not $2n$-Euclidean.

Theorem (McNicholl-Rute)

The following are equivalent:

1. $x \in \mathbb{R}^2$ is contained in a computable arc.
2. x is **1**-Euclidean (i.e., having a $\mathbb{R}$-tt-degree).

- In particular, the tt-degrees of points in a computable planar arc are exactly the **1**-Euclidean tt-degrees.
- A continuum is arc-like if it is an inverse limit of arcs.
- A pseudo-arc is a hereditarily indecomposable arc-like continuum.

Proposition

Let A be a computable pseudo-arc. Then,

$$D_{tt}(A) \cap \mathcal{D}_{tt}(\mathbb{R}) = \mathcal{D}_{tt}(2^\omega).$$

In particular, there are many points $x \in \mathbb{R}^2$ which are contained in a computable pseudo-arc, but not in any computable arc, and vice versa.

- Bing (1951): $\exists$ a hereditarily indecomposable n-dim. continuum.
- Hart, van Mill, and R. Pol's construction (2000) is effective. (Maybe Bing's original construction is also effective, but I didn't check)

Proposition

Let B_n be a computable hereditarily indec. n-dim. continuum.

$$\mathcal{D}_{tt}(B_{n+1}) \cap \mathcal{D}_{tt}(\mathbb{R}^{n+1}) \subseteq \{n\text{-dimensional } tt\text{-degrees}\}$$

Recall that x is n-dimensional and $y \equiv_{tt} x$ then y is also n-dimensional.

Proposition

Let x has a hyperimmune-free total T-degree. Then,

$$y \leq_T x \iff y \leq_{tt} x.$$

Proof

- Assume that $y \leq_T x$ via a total Φ on names.
- By totality, there is a Cauchy name p of x s.t. $p \leq_T x$.
- By h-immune-freeness, $\exists$ computable t s.t. $\Phi(p)(n)[t(n)]\downarrow$.
- Define $Q = \{q : (\forall n)\ \Phi(q)(n)[t(n)]\downarrow\}$, which is Π^0_1.
- Then we get a total computable Ψ such that $\Psi \upharpoonright Q = \Phi \upharpoonright Q$.
- Ψ can be modified to work with all names of x.

In this case, if x is n-dimensional and $y \equiv_T x$ then y is also n-dim.
However, every total T-degree contains a zero-dimensional point.
Thus, a h-immune-free total T-deg. consists of zero-dimensional points.

Proposition

Let x has a non-total T-degree.

$$y \text{ is finite dimensional} \implies [y \leq_T x \iff y \leq_{tt} x].$$

Proof

- Recall: y is n-dimensional $\Longrightarrow$ y is $(2n+1)$-Euclidean.
- Thus, one can assume that $y \in I^n$ for some $n \in \omega$.
- (J. Miller 04) A T-degree is total iff it is computably diagonalizable. Consider $\hat{x}(i,j,k) = x(i)b_j + b_k$ where b_e is the e-th rational. Clearly, $\hat{x} \equiv_{tt} x$. Miller showed (the contrapositive of) the following: If $x \in I^\omega$ is non-total then $\{z \in I : z \leq_T x\} = \{\hat{x}(n)\}_{n\in\omega^3}$
- Hence, $y \leq_T x \Longrightarrow y = (\hat{x}(a_0), \dots, \hat{x}(a_n))$ for some $a_0, \dots a_n$.
- Thus $y \leq_{tt} x$ via the total computable func. $z \mapsto (\hat{z}(a_0), \dots, \hat{z}(a_n))$.

If x is either hyperimmune-free or non-total, then

$$(\forall y)\ [\dim(y) < \infty \implies (y \leq_T x \iff y \leq_{tt} x)].$$

What happens if x is hyperimmune and total?

Proposition

Every hyperimmune total T-degree contains a point which is n-dimensional, but not $2n$-Euclidean.

Proof

- T: the set of all $\sigma \in 3^{2n+1}$ s.t. σ contains at most n many 1's.
- Let $(\sigma_k)_{k\in\omega} \in T^\omega$ be weakly 1-generic.
- Define $x_k = 0.\sigma_0(k)\sigma_1(k)\sigma_2(k)\sigma_3(k)\sigma_4(k)\dots$
- $(x_k)_{k<2n+1} \in \mathbb{R}^{2n+1}$ is in Menger's n-dim. universal compactum.
- Thus, $x = (x_k)_{k<2n+1}$ is n-dimensional.
- One can show that x is not $2n$-Euclidean.

Any h-immune total T-deg. contains a point which is n-dim., but not $2n$-Euclidean.

Corollary

Every hyperimmune total T-degree d contains infinitely many tt-degrees. Indeed, for any $n \in \omega$ there is $x_n \in d$ such that $\dim(x_n) = n$.

Let X be a computable compactum. For a T-degree d,

$$\mathcal{D}^d_{tt}(X) = \{\deg_{tt}(x) : x \in X \text{ and } \deg_T(x) = d\}.$$

We now consider the following:

$$\mathfrak{D}^d_{tt}(n\text{-dim}) = (\{\mathcal{D}^d_{tt}(X) : X \text{ is } n\text{-dim. computable compactum}\}, \subseteq).$$

Theorem

Let d be a hyperimmune total T-degree, and $n \in \omega$. Then, any countable upper semilattice embeds into $\mathfrak{D}^d_{tt}(n\text{-dim})$.

$$\mathcal{D}^d_{tt}(X) = \{\deg_{tt}(x) : x \in X \text{ and } \deg_T(x) = d\}.$$
$$\mathfrak{D}^d(n\text{-dim}) = (\{\mathcal{D}^d_{tt}(X) : X \text{ is } n\text{-dim. computable compactum}\}, \subseteq)$$

Any countable upper semilattice embeds into $\mathfrak{D}^d(n\text{-dim})$.

Idea of Proof: tt-reducibility relative to oracle
≈ first level Borel function ≈ continuous with finite mindchanges.

$$\frac{tt\text{-reducibility}}{T\text{-reducibility}} \approx \frac{\text{Medvedev-with-finite-mindchanges reducibility}}{\text{Muchnik reducibility}}$$

Want: every h-immune T-degree contains many kinds of tt-degrees.

- Medvedev-with-mindchanges reducibility was defined in my M.Sc thesis
- Higuchi-K. (2014): every nonzero Muchnik degree of a Π^0_1 class contains an infinite decreasing sequence of Medevedev^{mc} degrees.

A $\mathbf{\Pi^0_1}$ set $P \subseteq 2^\omega$ is special if it is nonempty and has no computable point.

Theorem (Higuchi-K. 2014)

For any special $\mathbf{\Pi^0_1}$ sets $P \subseteq 2^\omega$, there is a $\mathbf{\Pi^0_1}$ set $P^\blacktriangledown \subseteq 2^\omega$ s.t.

- P and $P^\blacktriangledown$ are Muchnik equivalent.
- P is not "Medvedev with finite mindchanges" reducible to $P^\blacktriangledown$.

Indeed, Higuchi-K. introduced the notion of hyperconcatenation $\blacktriangledown$ satisfying the following strong anticupping property:

$(\forall R)$ if P is Med^{mc}-reducible to $(Q \blacktriangledown P) \times R$ then P is Med^{mc}-reduc. to R.

Note that the idea of hyperconcatenation $\blacktriangledown$ comes from the proof of Simpson's embedding lemma (2003).

The hyperconcatenation $Q \blacktriangledown P$ is the mass problem s.t.

1. Try to answer the mass problem P with mindchanges.
2. Whenever your mind changes, give me **1** bit of an answer to Q.

A solution to the mass problem $Q \blacktriangledown P$ gives us

1. either an answer to P with finite mindchanges
2. or an answer to Q.

Properties of $Q \blacktriangledown P$:

- (Simpson 2003) $Q \blacktriangledown P$ is Muchnik equivalent to $Q \cup P$.
- (Higuchi-K. 2014) $(\forall R)\ P \leq^{mc}_{Med} (Q \blacktriangledown P) \times R \implies P \leq^{mc}_{Med} R$.

The hyperconcatenation $\blacktriangledown$ preserves the Muchnik-strength, but extremely decreases the Medvedev-with-finite-mindchanges strength!

- We can use ▼ to control the Medvedev^{mc} degree without affecting the Muchnik degree.
- Is there a construction similar to ▼ for controlling the tt-degree without affecting the T-degree and the topological dimension?

A construction similar to ▼ has been considered in topology:

V. A. Chatyrko, *Analogs of Cantor manifolds for transfinite dimensions*", Math. Notes Acad. Sci. USSR. **42** (1987)

The construction is "*the inverse limit of iterates of separable Alexandrov duplicates*."

What is the iterates of separable Alexandrov duplicates?

Let X be a compactum with $\mathbf{diam}(X) < 1$, and $(a_i)_{i>0}$ dense in X.

- Put a copy of X at height 0.
- Add a point a'_s above a_s at height 2^{-s}.
- Replace a'_s with a 2^{-s}-scaled copy X_s of X.
- Let a_{st} be the copy of a_t in X_s.
- Add a point a'_{st} above a_{st} at height 2^{-s-t}.
- Replace a'_{st} with a 2^{-s-t}-scaled copy X_{st} of X.
- Continue this procedure.

Think of the resulting space X^∞ as "X with mindchanges":
A Cauchy name of a point X^∞ is...

- Provide a name of a point in X with mindchanges.
- Whenever your mind changes, give me some a_s near your current position.

Note: X^∞ is a computable metric space whenever X is.

Compare two methods

The hyperconcatenation $Q \blacktriangledown P$ is...

1. Try to answer the mass problem P with mindchanges.
2. Whenever your mind changes, give me **1** bit of an answer to Q.

A Cauchy-name of a point in X^{∞} is...

1. Provide a name of a point in X with mindchanges.
2. Whenever your mind changes, give me some a_s near your current position.

Lemma (essentially by Chatyrko 1987)

- If X is n-dimensional, so is X^∞.
- X is second-level Borel isomorphic to X^∞.
- If X^∞ is first-level Borel isomorphic to Y^∞ then X is homeomorphic to Y.

Lemma

X, Y: computable metric spaces; d: hyperimmune total T-degree. If X does not embed into Y, then $\mathcal{D}^{d}_{tt}(X) \not\subseteq \mathcal{D}^{d}_{tt}(Y)$.

Theorem

Let d be a hyperimmune total T-degree, and $n \in \omega$. Then, any countable upper semilattice embeds into $\mathfrak{D}^{d}_{tt}(n\text{-}\mathbf{dim})$.

Conclusion

- The generalized T-degree theory has had great success.
- How about tt-degrees?

Conclusion

- The generalized $\boldsymbol{T}$-degree theory has had great success.
- How about $\boldsymbol{tt}$-degrees?
- Unfortunately, it is far from satisfactory at the present moment.
- Because computability plays very few roles in the proofs.
  It's merely a point-set topology!
- (Problem) Show a nice result about generalized $\boldsymbol{tt}$-degrees whose proof involves both computability and topology.