Finite Final Segments of the D.C.E. Turing Degrees

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Joint work with Yiqun Liu, Yong Liu, Cheng Peng, and Yue Yang
(National University of Singapore)
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(Nanyang Technological University)

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Definition

A set $A \subseteq \omega$ is a *d.c.e. set (a difference of c.e. sets)* if there is a computable approximation $\{A_s\}_{s \in \omega}$ with $A_0 = \emptyset$, $A = \lim_s A_s$, and for all x, $|\{s \mid A_{s+1}(x) \neq A_s(x)\}| \leq 2$.

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D.C.E. Nondensity Theorem (Cooper, Harrington, Lachlan, Lempp, Soare 1991)

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Question

Which other finite lattices can be embedded as final segments into the d.c.e. Turing degrees?

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Every finite Boolean algebra is embeddable into the d.c.e. degrees as a final segment.

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Theorem 1

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Theorem 2

The 3-element chain $\{0 < c < 1\}$ is embeddable into the d.c.e. degrees as a final segment.

Conjecture 1

Every finite distributive lattice is embeddable into the d.c.e. degrees as a final segment.

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Every finite interval dismantlable lattice is embeddable into the d.c.e. degrees as a final segment.

Here, a finite lattice L is *interval dismantlable* if there is a finite binary tree T such that

- the root λ is associated with the L-interval $[0, 1]$, and
- each note $\sigma \in T$ associated with an interval $[c, d]$, say,
  - is a leaf if $[c, d]$ has only one element; or
  - has two successors $\sigma^\frown\langle 0\rangle$ and $\sigma^\frown\langle 1\rangle$ associated with nonempty L-subintervals $[c, d']$ and $[c', d]$, resp., partitioning $[c, d]$.

We build a d.c.e. set E and a c.e. set A and ensure the following requirements, for all d.c.e. sets U and Turing functionals Ψ:

$$\mathcal{S}_U : K = \Gamma(U \oplus E) \text{ or } U = \Delta(E)$$
$$\mathcal{R}_\Psi : A \neq \Psi(E)$$

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The typical conflict is between an $\mathcal{R}$-strategy below an $\mathcal{S}$-strategy building its functional Γ: Enumerating a diagonalization witness x into A and trying to restrain E to preserve $\Psi(E; x) = 0$ can trigger a number y entering K and requiring Γ-correction via E unless U changes.

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But then $\Psi(E; x) = 0$ will be destroyed iff $\Gamma(U \oplus E; y)$ needs to be corrected via E iff $\Gamma(U \oplus E; y)$ is not corrected by a U-change iff U can be computed by E via a functional Δ (up to the use of $\Gamma(U \oplus E; y)$).

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Now iterate and nest.

We build d.c.e. sets E, D_0 and D_1 and a c.e. set A and ensure the following requirements, for all d.c.e. sets U and functionals Ψ:

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(A later $\mathcal{R}_{\Psi,\lambda}$-strategy may kill Γ_λ and $\Delta_{\langle 0\rangle}$ and build Δ_λ.)
The new feature is the conflict between toggling D against U-changes, and keeping diagonalization witnesses in D; with two $\mathcal{S}$-requirements above, this causes a serious conflict that requires multiple attempts at building Γ's and Δ's.

Thanks!