Who asked *us*?

How the theory of computing answers questions that weren't about computing

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Oberwolfach
Computability Theory
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Algorithmic Information (Kolmogorov Complexity)

The Kolmogorov complexity of a string $x \in \{0,1\}^*$ is

$$K(x) = \min\left\{|\pi| \,\middle|\, \pi \in \{0,1\}^* \text{ and } U(\pi) = x\right\},$$

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- x is “random” if $K(x) \approx |x|$.
- Routine coding extends this to $K(x)$ for $x \in \mathbb{N}$, $x \in \mathbb{Q}$, $x \in \mathbb{Q}^n$, etc.

Dimensions of Points

Work in Euclidean space $\mathbb{R}^n$.

The Kolmogorov complexity of a set $E \subseteq \mathbb{Q}^n$ is

$$K(E) = \min\{K(q) \mid q \in E\}\,.$$

(Shen and Vereschagin 2002)

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Note that

$$E \subseteq F \Rightarrow K(E) \geq K(F)\,.$$

Dimensions of Points

Let $x \in \mathbb{R}^n$ and $r \in \mathbb{N}$. The Kolmogorov complexity of x at precision r is

$$K_r(x) = K\left(B_{2^{-r}}(x)\right),$$

i.e., the number of bits required to specify some rational point $q \in \mathbb{Q}^n$ such that $|q - x| \leq 2^{-r}$.

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Easy fact. $0 \leq \dim(x) \leq n$, and there are uncountably many points of each dimension in this interval.

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Old fact (J. Lutz '00 + Hitchcock '03). If $E \subseteq \mathbb{R}^n$ is a union of Π^0_1 sets, then

$$\dim_H(E) = \sup_{x \in E} \dim(x) .$$

classical Hausdorff (fractal) dimension

dimensions of individual points

$\therefore$ Dimensions of points are geometrically meaningful.

Dimensions of Points

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$\dim(x)$ is the "Σ^0_1 version" of $\dim_H$. (Hausdorff dimension)

$\mathrm{Dim}(x)$ is the "Σ^0_1 version" of $\dim_P$. (packing dimension)

Point-to-Set Principle

Theorem (J. Lutz and N. Lutz, STACS '17)

For every $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x)\,.$$

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$\therefore$ In order to prove a lower bound

$$\dim_H(E) \geq \alpha\,,$$

it suffices to show that

$$(\forall A \subseteq \mathbb{N})(\forall \varepsilon > 0)(\exists x \in E)\, \dim^A(x) \geq \alpha - \varepsilon$$

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or, if you're lucky, that

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For every $E \subseteq \mathbb{R}^n$,

$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \mathrm{Dim}^A(x)\,.$$

Kakeya Sets in $\mathbb{R}^2$

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Kakeya Conjecture. Every Kakeya set in $\mathbb{R}^n$ has Hausdorff dimension n.

- An important open problem for $n \geq 3$.

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It doesn't seem to help in $\mathbb{R}^n$ for $n \geq 3$.

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$$\dim(x, mx + b) \geq \dim^{m,b}(x) + \min\{\dim(m, b), \dim^{m,b}(x)\} .$$

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Corollary. For every $m, b \in \mathbb{R}$ there exist $x_1, x_2 \in \mathbb{R}$ such that

$$\dim(x_1, mx_1 + b) - \dim(x_2, mx_2 + b) \geq 1.$$

$\therefore$ The answer to the above question is "No!"

Generalized Furstenberg sets

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For $\alpha \in (0, 1]$, a set $E \subseteq \mathbb{R}^2$ is α-Furstenberg if, for every $e \in S^1$ ($=$ the unit circle in $\mathbb{R}^2$), there is a line $\mathcal{L}_e$ in direction e such that $\dim_H(\mathcal{L}_e \cap E) \geq \alpha$.

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Definition (Molter and Rela 2012)

For $\alpha, \beta \in (0, 1]$, a set $E \subseteq \mathbb{R}^2$ is (α, β)-generalized Furstenberg if there is a set $J \subseteq S^1$ such that $\dim_H(J) \geq \beta$ and, for every $e \in J$, there is a line $\mathcal{L}_e$ in direction e such that $\dim_H(\mathcal{L}_e \cap E) \geq \alpha$.

Generalized Furstenberg sets

Theorem (probably Furstenberg and Katznelson)

Fore $\alpha \in (0, 1]$, every α-Furstenberg set $E \subseteq \mathbb{R}^2$ satisfies

$$\dim_H(E) \geq \alpha + \max\left\{\frac{1}{2}, \alpha\right\} .$$

Note that Davies's theorem follows from the case $\alpha = 1$.

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Theorem (N. Lutz and D. Stull, TAMC '17)

For $\alpha, \beta \in (0, 1]$, *every* (α, β)*-generalized Furstenberg set* $E \subseteq \mathbb{R}^2$ *satisfies*

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Note that this improves on the theorem of Molter and Rela exactly when $\alpha < 1$, $\beta < 1$, and $\beta < 2\alpha$. Hence it doesn't improve the bound on α-Furstenberg sets.

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It is the first use of algorithmic fractal dimensions to prove a new theorem in classical fractal geometry!

Intersections and Products of Fractals

The following are fundamental, nontrivial, textbook theorems of fractal geometry.

Product Formula (Marstrand 1954). For all sets $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$,

$$\dim_H(E \times F) \geq \dim_H(E) + \dim_H(F).$$

Intersection Formula (Kahane 1986; Mattila 1984, 1985). For all Borel sets $E, F \subseteq \mathbb{R}^n$ and almost every $z \in \mathbb{R}^n$,

$$\dim_H(E \cap (F + z)) \leq \max\{0, \dim_H(E \times F) - n\}.$$

Note: The product formula was known earlier with extra assumptions on E and F. Marstrand deployed nontrivial machinery to prove it for arbitrary sets. Textbooks usually just prove it for Borel sets.

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This paper also uses a similar method to give a much simpler proof of the general Product Formula, along with analogous results for packing dimension.

Pointwise Dimensions

Classical fractal geometry has a pointwise notion of dimension.

An outer measure on $\mathbb{R}^n$ is a function $\nu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ satisfying

- $\nu(\emptyset) = 0$,
- $E \subseteq F \;\Rightarrow\; \nu(E) \leq \nu(F)$, and
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An outer measure ν on $\mathbb{R}^n$ is

- finite if $\nu(\mathbb{R}^n) < \infty$, and
- locally finite if every $x \in \mathbb{R}^n$ has a neighborhood N with $\nu(N) < \infty$.

Pointwise Dimensions

Definition

Let ν be a locally finite outer measure on $\mathbb{R}^n$, and let $x \in \mathbb{R}^n$. The lower and upper pointwise dimensions of ν at x are

$$\dim_\nu(x) = \liminf_{r\to\infty} \frac{\log \frac{1}{\nu(B_{2^{-r}}(x))}}{r}$$

and

$$\mathrm{Dim}_\nu(x) = \limsup_{r\to\infty} \frac{\log \frac{1}{\nu(B_{2^{-r}}(x))}}{r},$$

respectively.

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Are these in any way related to the algorithmic dimensions $\dim(x)$ and $\mathrm{Dim}(x)$?

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For each $E \subseteq \mathbb{R}^n$, let

$$\kappa(E) = 2^{-K(E)} .$$

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1. *κ is a finite outer measure on $\mathbb{R}^n$.*

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2. *For all $x \in \mathbb{R}^n$, $\dim(x) = \dim_\kappa(x)$ and $\mathrm{Dim}(x) = \mathrm{Dim}_\kappa(x)$.*
3. *This relativizes and interacts informatively with the Point-to-Set Principle.*

Thank you!