Computability and incomputability of projection functions in Euclidean space

Alberto Marcone
(joint work with Guido Gherardi)

Dipartimento di Scienze Matematiche, Informatiche e Fisiche
Università di Udine
Italy
alberto.marcone@uniud.it
http://www.dimi.uniud.it/marcone

Computability Theory Workshop
January 8–12, 2018
Oberwolfach

Projecting in Euclidean space

Projecting a point over a non-empty subset of the Euclidean space is an operation deeply grounded in our geometrical intuition of the spatial continuum, and has many important applications in mathematics.

Projecting in Euclidean space

Projecting a point over a non-empty subset of the Euclidean space is an operation deeply grounded in our geometrical intuition of the spatial continuum, and has many important applications in mathematics.

Given $x \in \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$ with $A \neq \emptyset$ we want to find $y \in A$ such that $d(x, y) = d(x, A)$.

When A is closed y does exist, although in general is not unique.

Projecting in Euclidean space

Projecting a point over a non-empty subset of the Euclidean space is an operation deeply grounded in our geometrical intuition of the spatial continuum, and has many important applications in mathematics.

Given $x \in \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$ with $A \neq \emptyset$ we want to find $y \in A$ such that $d(x, y) = d(x, A)$.

When A is closed y does exist, although in general is not unique.

Does the intuitive, even empirical, naturalness of this problem correspond to an algorithmic simplicity of solution?

1. **Computing with reals and closed sets**

2. **Weihrauch reducibility**

3. **Projection operators**

Computable partial functions from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}^\mathbb{N}$

A partial function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is computable if there is an oracle Turing machine T such that for every $p \in \mathrm{dom}(F)$ the function computed by T with oracle p is total and coincides with $F(p)$.

These are also known as Lachlan functionals.

Computable partial functions from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}^\mathbb{N}$

A partial function $F : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is computable if there is an oracle Turing machine T such that for every $p \in \mathrm{dom}(F)$ the function computed by T with oracle p is total and coincides with $F(p)$.

These are also known as Lachlan functionals.

Notice that every computable partial function is continuous.

Represented spaces

A representation σ_X of a set X is a surjective partial function $\sigma_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$.

The pair (X, σ_X) is a represented space.

If $x \in X$ a σ_X-name for x is any $p \in \mathbb{N}^{\mathbb{N}}$ such that $\sigma_X(p) = x$.

Represented spaces

A representation σ_X of a set X is a surjective partial function $\sigma_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$.

The pair (X, σ_X) is a represented space.

If $x \in X$ a σ_X-name for x is any $p \in \mathbb{N}^{\mathbb{N}}$ such that $\sigma_X(p) = x$.

$x \in X$ is computable (w.r.t. σ_X) if it has a computable σ_X-name.

Represented spaces

A representation σ_X of a set X is a surjective partial function $\sigma_X : \subseteq \mathbb{N}^{\mathbb{N}} \to X$.

The pair (X, σ_X) is a represented space.

If $x \in X$ a σ_X-name for x is any $p \in \mathbb{N}^{\mathbb{N}}$ such that $\sigma_X(p) = x$.

$x \in X$ is computable (w.r.t. σ_X) if it has a computable σ_X-name.

Representations are analogous to the codings used in reverse mathematics to speak about various mathematical objects in subsystems of second order arithmetic.

Computable partial functions between represented spaces

If (X, σ_X) and (Y, σ_Y) are represented spaces and $f : \subseteq X \rightrightarrows Y$ we say that f is computable if there exists a computable $F : \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\sigma_Y(F(p)) \in f(\sigma_X(p))$ whenever $f(\sigma_X(p))$ is defined, i.e. p is a name for an element of $\mathrm{dom}(f)$.

Such an F is called a realizer of f.

Computable partial functions between represented spaces

If (X, σ_X) and (Y, σ_Y) are represented spaces and $f : \subseteq X \rightrightarrows Y$ we say that f is computable if there exists a computable $F : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $\sigma_Y(F(p)) \in f(\sigma_X(p))$ whenever $f(\sigma_X(p))$ is defined, i.e. p is a name for an element of $\mathrm{dom}(f)$. Such an F is called a realizer of f.

Notice that different names of the same $x \in \mathrm{dom}(f)$ might be mapped by F to names of different elements of $f(x)$.

Some examples of representations

Let (X, D, d) be a computable metric space.

Some examples of representations

Let (X, D, d) be a computable metric space.

Cauchy representation of X: $p \in \mathbb{N}^{\mathbb{N}}$ is a name for $x \in X$ if p lists a sequence $(x_i) \subseteq D$ such that $d(x_i, x_{i+1}) \leq 2^{-i}$ for every i and $\lim x_i = x$.

Some examples of representations

Let (X, D, d) be a computable metric space.

Cauchy representation of X: $p \in \mathbb{N}^{\mathbb{N}}$ is a name for $x \in X$ if p lists a sequence $(x_i) \subseteq D$ such that $d(x_i, x_{i+1}) \leq 2^{-i}$ for every i and $\lim x_i = x$.

negative representation of the set $\mathcal{A}_-(X)$ of closed subsets of X: $p \in \mathbb{N}^{\mathbb{N}}$ is a name for the closed set A if p lists a sequence of open balls with center in D and rational radius whose union is $X \setminus A$.

Some examples of representations

Let (X, D, d) be a computable metric space.

Cauchy representation of X: $p \in \mathbb{N}^{\mathbb{N}}$ is a name for $x \in X$ if p lists a sequence $(x_i) \subseteq D$ such that $d(x_i, x_{i+1}) \leq 2^{-i}$ for every i and $\lim x_i = x$.

negative representation of the set $\mathcal{A}_-(X)$ of closed subsets of X: $p \in \mathbb{N}^{\mathbb{N}}$ is a name for the closed set A if p lists a sequence of open balls with center in D and rational radius whose union is $X \setminus A$.

positive representation of the set $\mathcal{A}_+(X)$ of closed subsets of X: $p \in \mathbb{N}^{\mathbb{N}}$ is a name for the closed set A if p lists a dense sequence in A.

Some examples of representations

Let (X, D, d) be a computable metric space.

Cauchy representation of X: $p \in \mathbb{N}^{\mathbb{N}}$ is a name for $x \in X$ if p lists a sequence $(x_i) \subseteq D$ such that $d(x_i, x_{i+1}) \leq 2^{-i}$ for every i and $\lim x_i = x$.

negative representation of the set $\mathcal{A}_{-}(X)$ of closed subsets of X: $p \in \mathbb{N}^{\mathbb{N}}$ is a name for the closed set A if p lists a sequence of open balls with center in D and rational radius whose union is $X \setminus A$.

positive representation of the set $\mathcal{A}_{+}(X)$ of closed subsets of X: $p \in \mathbb{N}^{\mathbb{N}}$ is a name for the closed set A if p lists a dense sequence in A.

total representation of the set $\mathcal{A}(X)$ of closed subsets of X: $p = p_0 \oplus p_1 \in \mathbb{N}^{\mathbb{N}}$ is a name for the closed set A if p_0 is a name for $A \in \mathcal{A}_{-}(X)$ and p_1 is a name for $A \in \mathcal{A}_{+}(X)$.

Representations of closed sets and distance

Let (X, D, d) be a computable metric space.

If $A \subseteq X$ is closed is important for us to compute the distance function $x \mapsto d(x, A)$.

Representations of closed sets and distance

Let (X, D, d) be a computable metric space.

If $A \subseteq X$ is closed is important for us to compute the distance function $x \mapsto d(x, A)$.

- if $A \in \mathcal{A}_{-}(X)$ then we can approximate $d(x, A)$ from below;

Representations of closed sets and distance

Let (X, D, d) be a computable metric space.

If $A \subseteq X$ is closed is important for us to compute the distance function $x \mapsto d(x, A)$.

- if $A \in \mathcal{A}_-(X)$ then we can approximate $d(x, A)$ from below;
- if $A \in \mathcal{A}_+(X)$ then we can approximate $d(x, A)$ from above;

Representations of closed sets and distance

Let (X, D, d) be a computable metric space.

If $A \subseteq X$ is closed is important for us to compute the distance function $x \mapsto d(x, A)$.

- if $A \in \mathcal{A}_-(X)$ then we can approximate $d(x, A)$ from below;
- if $A \in \mathcal{A}_+(X)$ then we can approximate $d(x, A)$ from above;
- if $A \in \mathcal{A}(X)$ then we can compute $d(x, A)$.

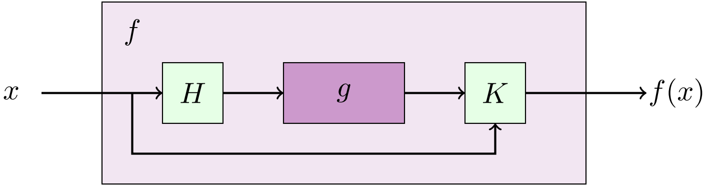
Weihrauch reducibility

Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be partial multi-valued functions between represented spaces.

Weihrauch reducibility

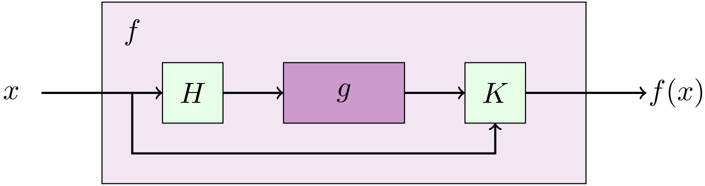
Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be partial multi-valued functions between represented spaces.

f is Weihrauch reducible to g, $f \leq_{\mathrm{W}} g$, if there are computable $H :\subseteq X \rightrightarrows Z$ and $K :\subseteq X \times W \rightrightarrows Y$ such that $K(x, gH(x)) \subseteq f(x)$ for all $x \in \mathrm{dom}(f)$:



In other words, for all $x \in \mathrm{dom}(f)$, we have $H(x) \subseteq \mathrm{dom}(g)$ and $K(x, w) \in f(x)$ for every $w \in g(H(x))$.

Weihrauch reducibility



$f \leq_{\mathrm{W}} g$ means that the problem of computing f can be computably and uniformly solved by using in each instance a single computation of g:

H modifies the input of f to feed it to g, while K, using also the original input, transforms the output of g into the correct output of f.

The Weihrauch lattice

$\leq_W$ is reflexive and transitive and induces the equivalence relation $\equiv_W$.

The $\equiv_W$-equivalence classes are called Weihrauch degrees.

The partial order on the sets of Weihrauch degrees is a distributive bounded lattice with several natural and useful algebraic operations: the Weihrauch lattice.

Some milestones in the Weihrauch lattice

$\lim :\subseteq (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ maps a convergent sequence in Baire space to its limit, and corresponds to $0'$.

Some milestones in the Weihrauch lattice

$\mathsf{lim} :\subseteq (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ maps a convergent sequence in Baire space to its limit, and corresponds to $0'$.

$\mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X$ is the choice function for the computable metric space X: it picks from a nonempty closed set in X one of its elements.

Some milestones in the Weihrauch lattice

$\mathsf{lim} : \subseteq (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ maps a convergent sequence in Baire space to its limit, and corresponds to $0'$.

$\mathsf{C}_X : \subseteq \mathcal{A}_-(X) \rightrightarrows X$ is the choice function for the computable metric space X: it picks from a nonempty closed set in X one of its elements.

$\mathsf{BWT}_X : \subseteq X^{\mathbb{N}} \rightrightarrows X$ for X a computable metric space is the multi-valued function that associates to a sequence (x_n) with compact closure the set of its cluster points.

Some milestones in the Weihrauch lattice

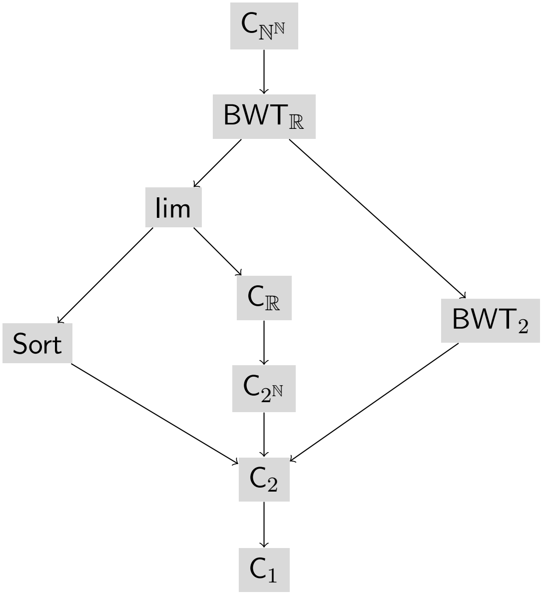
$\mathsf{lim} :\subseteq (\mathbb{N}^\mathbb{N})^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ maps a convergent sequence in Baire space to its limit, and corresponds to $0'$.

$\mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X$ is the choice function for the computable metric space X: it picks from a nonempty closed set in X one of its elements.

$\mathsf{BWT}_X :\subseteq X^\mathbb{N} \rightrightarrows X$ for X a computable metric space is the multi-valued function that associates to a sequence (x_n) with compact closure the set of its cluster points.

$\mathsf{Sort} : 2^\mathbb{N} \to 2^\mathbb{N}$ maps p to $0^n 1^\omega$ if $|\{ i \mid p(i) = 0 \}| = n$ and to 0^ω if p has infinitely many 0's.

A picture


$C_{\mathbb{N}^{\mathbb{N}}}$
$BWT_{\mathbb{R}}$
lim
$C_{\mathbb{R}}$
BWT_2
Sort
$C_{2^{\mathbb{N}}}$
C_2
C_1

Exact projections

Let X be a computable metric space.
The (exact) negative, positive and total projection operators on X are the partial multi-valued functions Proj_X^-, Proj_X^+ and Proj_X which associate to every $x \in X$ (with Cauchy representation) and every closed $A \neq \emptyset$ (with negative, positive and total representation, respectively) the set

$$\{ y \in A \mid d(x, y) = d(x, A) \}.$$

Exact projections

Let X be a computable metric space.
The (exact) negative, positive and total projection operators on X are the partial multi-valued functions Proj^-_X, Proj^+_X and Proj_X which associate to every $x \in X$ (with Cauchy representation) and every closed $A \neq \emptyset$ (with negative, positive and total representation, respectively) the set

$$\{\, y \in A \mid d(x,y) = d(x,A) \,\}.$$

$$\begin{aligned}
\mathsf{Proj}^-_X &:\subseteq X \times \mathcal{A}_-(X) \rightrightarrows X,\\
\mathsf{Proj}^+_X &:\subseteq X \times \mathcal{A}_+(X) \rightrightarrows X,\\
\mathsf{Proj}_X &:\subseteq X \times \mathcal{A}(X) \rightrightarrows X
\end{aligned}$$

Approximate projections

Let X be a computable metric space and fix $\varepsilon > 0$.
The ε-approximate negative, positive and total projection operators on X
are the partial multi-valued functions $\varepsilon\text{-}\operatorname{Proj}_X^-$, $\varepsilon\text{-}\operatorname{Proj}_X^+$ and $\varepsilon\text{-}\operatorname{Proj}_X$ which associate to every $x \in X$ (with Cauchy representation) and every closed $A \neq \emptyset$ (with negative, positive and total representation, respectively) the set

$$\{ y \in A \mid d(x, y) \leq (1 + \varepsilon) d(x, A) \}.$$

Approximate projections

Let X be a computable metric space and fix $\varepsilon > 0$.
The ε-approximate negative, positive and total projection operators on X
are the partial multi-valued functions $\varepsilon\text{-}\mathrm{Proj}_X^-$, $\varepsilon\text{-}\mathrm{Proj}_X^+$ and $\varepsilon\text{-}\mathrm{Proj}_X$ which associate to every $x \in X$ (with Cauchy representation) and every closed $A \neq \emptyset$ (with negative, positive and total representation, respectively) the set

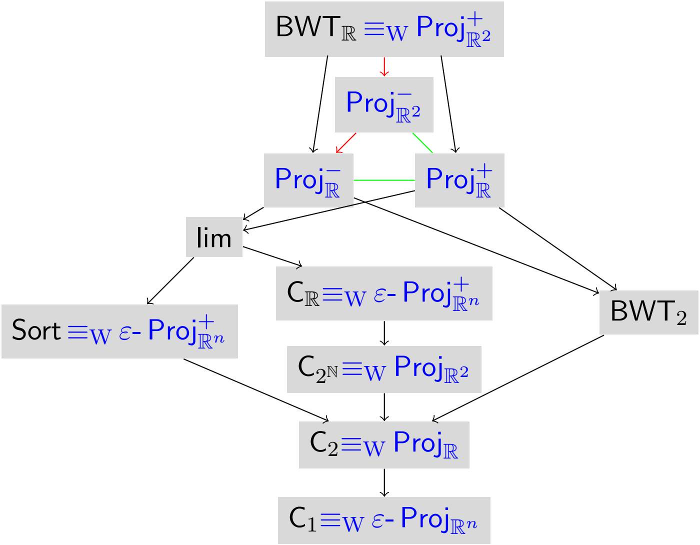
$$\{\, y \in A \mid d(x,y) \leq (1+\varepsilon) d(x,A) \,\}.$$

$$\varepsilon\text{-}\mathrm{Proj}_X^- :\subseteq X \times \mathcal{A}_-(X) \rightrightarrows X,$$
$$\varepsilon\text{-}\mathrm{Proj}_X^+ :\subseteq X \times \mathcal{A}_+(X) \rightrightarrows X,$$
$$\varepsilon\text{-}\mathrm{Proj}_X :\subseteq X \times \mathcal{A}(X) \rightrightarrows X$$

Summary of results

Proj.	Repr.	Dim.	Weihrauch degree
Exact	negative	$n = 1$	$>_W$ lim, $>_W$ BWT_2, $<_W$ $\mathsf{BWT}_{\mathbb{R}}$
		$n \geq 2$	$>_W$ lim, $>_W$ BWT_2, $\leq_W$ $\mathsf{BWT}_{\mathbb{R}}$
	positive	$n = 1$	$>_W$ lim, $>_W$ BWT_2, $<_W$ $\mathsf{BWT}_{\mathbb{R}}$
		$n \geq 2$	$\equiv_W$ $\mathsf{BWT}_{\mathbb{R}}$
	total	$n = 1$	$\equiv_W$ C_2
		$n \geq 2$	$\equiv_W$ $\mathsf{C}_{2^{\mathbb{N}}}$
Approx.	negative	$n \geq 1$	$\equiv_W$ $\mathsf{C}_{\mathbb{R}}$
	positive	$n \geq 1$	$\equiv_W$ Sort
	total	$n \geq 1$	computable

The projections picture



Thank you for your attention!

LOGIC COLLOQUIUM

July 23–28
Udine, Italy 2018

http://lc18.uniud.it/