A point-to-set principle for separable metric spaces

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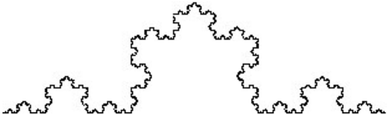
Today

- Hausdorff dimension
- Lutz's effective dimension for Cantor and Euclidean spaces
- Effective dimension in separable spaces
- Point-to-set principle

Hausdorff definition of dimension

Hausdorff, 1919: Rigorous formulation of dimension.



Hausdorff definition of dimension

Let ρ be a metric on a set X.

- The diameter of a set $E \subseteq X$ is

$$\mathrm{diam}(E) = \sup\{\rho(x,y) \,|\, x, y \in E\}.$$

- For $E \subseteq X$ and $\delta > 0$, a δ-cover of E is a collection $\mathcal{U}$ such that for all $U \in \mathcal{U}$, $\mathrm{diam}(U) \le \delta$ and

$$E \subseteq \bigcup_{U \in \mathcal{U}} U.$$

- For $s \ge 0$,
  $H^s(E) = \lim_{\delta \to 0} \inf_{\mathcal{U} \text{ is a } \delta\text{-cover of } E} \sum_{U \in \mathcal{U}} \mathrm{diam}(U)^s$

$H^s(E) =$ the s-dimensional Hausdorff measure of E

Hausdorff definition of dimension

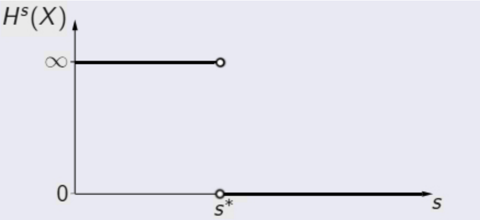
$H^s(E) = \lim_{\delta \to 0} \inf_{\mathcal{U} \text{ is a } \delta\text{-cover of } E} \sum_{U \in \mathcal{U}} \mathrm{diam}(U)^s$

Definition (Fractal Dimension)

Let ρ be a metric on X, and let $E \subseteq X$.

- (Hausdorff 1919) The Hausdorff dimension of E is

$$\dim_{\mathrm{H}}(E) = \inf \{ s \,|\, H^s(E) = 0 \} .$$



For Cantor space

- $\{0,1\}^\infty$ is the set of infinite binary sequences
- For $x \in \{0,1\}^\infty$, $x \upharpoonright n$ the the length n finite prefix of x

Constructive dimension in Cantor space

Definition

For every $x \in \{0,1\}^\infty$, $E \subseteq \{0,1\}^\infty$,

$$\mathrm{cdim}(x) = \liminf_{n} \frac{\mathrm{K}(x \upharpoonright n)}{n}.$$

$$\mathrm{cdim}(E) = \sup_{x \in E} \mathrm{cdim}(x).$$

- For a finite string w, $\mathrm{K}(w)$ is the length of the shortest description from which w can be computably recovered.
- Lutz original definition of constructive dimension uses gambling, this is a characterization.

Effective dimension in Cantor and Euclidean spaces

- Very robust concepts, they can be defined using
  - measure theory
  - gambling
  - information theory
- Resource-bounded versions are natural and useful
- It is non necessarily zero and meaningful on singletons.
- By absolute stability effective dimension can be investigated in terms of the dimension of individual points.
- For Σ^0_2 sets, constructive dimension is exactly Hausdorff dimension ... For a while this was good enough

How effective dimension can be used for classical geometry

Classical dimension can be characterized in terms of effective dimension (point-to-set principle)

Theorem (J.Lutz, N.Lutz 2017)

For every $E \subseteq \{0,1\}^\infty$,

$$\dim_H(E) = \min_{B \subseteq \{0,1\}^*} \operatorname{cdim}^B(E).$$

- **This theorem allows us to prove classical dimension results using Kolmogorov complexity**, already two very interesting ones (N.Lutz-Stull on generalized Furstenberg sets, N. Lutz on the intersection formula)
- We can now investigate the dimension of a set in terms of the dimension of its points
- We want to use it in other spaces

Kolmogorov complexity in a separable space

- Let (X, ρ) be a separable metric space. Let D be a countable dense set and $f : \{0,1\}^* \to D$ be surjective
- What is the information content of $x \in X$?

Definition

Let $x \in X, r \in \mathbb{N}$. The Kolmogorov complexity of x at precision r is

$$\mathrm{K}_r^f(x) = \inf \left\{ \mathrm{K}(w) \,\middle|\, \rho(x, f(w)) \leq 2^{-r} \right\}.$$

Constructive dimension in a separable space

(X, ρ) is a separable metric space, D is a countable dense set, and $f : \{0,1\}^* \to D$ surjective

Definition

Let $x \in X$,

$$\mathrm{cdim}^f(x) = \liminf_r \frac{\mathrm{K}_r^f(x)}{r}.$$

Let $E \subseteq X$,

$$\mathrm{cdim}^f(E) = \sup_{x \in E} \mathrm{cdim}^f(x).$$

Both definitions relativize to any oracle B by using $\mathrm{K}^B(w)$

Constructive dimension in a separable space

- In this case the effectivization is done from information content
- It can be done from measure
- For spaces with a suitable regularity condition it can be done through gambling (pretty useful for resource-bounds)
- Similarly **packing dimension** and **exact dimension** can be effectivized for all separable spaces
- Notice that in particular X can have ∞ Hausdorff dimension

Point to set principle for separable X

Theorem (Main result)

Let $E \subseteq X$. Then

$$\dim_{\mathrm{H}}(E) = \min_{B \subseteq \{0,1\}^*} \mathrm{cdim}^{f,B}(E).$$

Proof $\mathrm{cdim}^{f,B}(E) \leq \dim_{\mathrm{H}}(E)$

Observation: In the definition of Hausdorff measure we can restrict covers to balls with center in D and rational radius.

- $E \subseteq X$, $d = \dim_{\mathrm{H}}(E)$, $s > d$, $s \in \mathbb{Q}$
- Then $H^s(A) = 0$
- for every $t \in \mathbb{N}$: there is

$$\left\{ d_i^{t,s} \,|\, i \in \mathbb{N} \right\} \subseteq D, \left\{ q_i^{t,s} \,|\, i \in \mathbb{N} \right\} \subseteq \mathbb{Q}$$

  - $\left\{ B(d_i^{t,s}, q_i^{t,s}) \,|\, i \in \mathbb{N} \right\}$ is a 2^{-t} cover of E
  - $\sum_i (q_i^{t,s})^s < 1$.

Proof $\mathrm{cdim}^{f,B}(E) \leq \dim_{\mathrm{H}}(E)$

We codify all covers in an oracle:

- using $f : \{0,1\}^* \to D$,

$$h(i,t,s) = (w_i^{t,s}, q_i^{t,s})$$

  with $f(w_i^{t,s}) = d_i^{t,s}$
- Let B be an oracle encoding h (B encodes the covers)
- We will show that for any $x \in E$, $\mathrm{cdim}^{f,B}(x) \leq s$

Proof $\mathrm{cdim}^{f,B}(E) \leq \dim_{\mathrm{H}}(E)$

- Fix $x \in E$, $t \in \mathbb{N}$. Let i be such that $x \in B(d_i^{t,s}, q_i^{t,s})$.
- Let r be such that $2^{-(r+1)} < q_i^{t,s} \leq 2^{-r}$.

$$\mathrm{K}_r^{f,B}(x) \leq \mathrm{K}^B(i,t,s)$$

  - $t \leq r+1$
  - Since $\sum_i (q_i^{t,s})^s < 1$ there are less than $2^{(r+1)s}$ values of i for which $q_i^{t,s} > 2^{-(r+1)}$
- Therefore $\mathrm{K}^B(i,t,s) \leq (r+1)s + \log(r+1) + O(1)$
- for infinitely many r, $\mathrm{K}_r^{f,B}(x) \leq (r+1)s + \log(r+1) + O(1)$

Proof $\mathrm{cdim}^{f,B}(E) \leq \dim_{\mathrm{H}}(E)$

- for infinitely many r, $\mathrm{K}_r^{f,B}(x) \leq (r+1)s + \log(r+1) + O(1)$

$$\mathrm{cdim}^{f,B}(x) = \liminf_r \frac{\mathrm{K}_r^{f,B}(x)}{r} \leq s.$$

$$\mathrm{cdim}^{f,B}(E) \leq \dim_{\mathrm{H}}(E)$$

Proof $\dim_{\mathrm{H}}(E) \leq \mathrm{cdim}^{f,B}(E)$

For the other direction

- let $x \in E$, let s, s' be such that $\mathrm{cdim}^{f,B}(x) < s' < s$.
- Then i.o. r, $\mathrm{K}_r^{f,B}(x) < s'r$.
- Let

$$\mathcal{U}_r = \left\{ B(f(w), 2^{-r}) \,\middle|\, \mathrm{K}(w) \leq s'r \right\},$$
$$\mathcal{W}_r = \cup_{k=r}^{\infty} \mathcal{U}_k$$

- there are infinitely many r for which $x \in \mathcal{U}_r$
- for every r, $x \in \mathcal{W}_r$, and $\mathcal{W}_r$ is a 2^{-r}-cover of E with

$$\sum_{B(f(w),2^{-k}) \in \mathcal{W}_r} 2^{-ks} = \sum_{k=r}^{\infty} \sum_{B(f(w),2^{-k}) \in \mathcal{U}_k} 2^{-ks}$$
$$< \sum_{k=r}^{\infty} 2^{s'k+1} 2^{-ks} < \infty$$

Proof $\dim_{\mathrm{H}}(E) \leq \mathrm{cdim}^{f,B}(E)$

- So $\mathcal{W}_r$ witnesses that $\dim_{\mathrm{H}}(E) \leq s$ and we conclude that $\dim_{\mathrm{H}}(E) \leq \mathrm{cdim}^{f,B}(x)$

$$\dim_{\mathrm{H}}(E) \leq \mathrm{cdim}^{f,B}(E)$$

Conclusions

- Lutz effectivization of Hausdorff dimension can be generalized all separable metric spaces via Kolmogorov complexity
- The point-to-set principle allows us to capture classical Hausdorff dimension through the pointwise analysis of the dimension of sets
- Let us use it to solve open problems in fractal geometry

Exact dimension: Kolmogorov complexity characterization

Let g be increasing in both arguments. For $s \geq 0$,

$$H^{g,s}(E) = \lim_{\delta \to 0} \inf_{\mathcal{U} \text{ is a } \delta\text{-cover of } E} \sum_{U \in \mathcal{U}} g(s, \mathrm{diam}(U))$$

$$\dim_{\mathrm{H}}^{(g)}(E) = \inf \{ s \,|\, H^{g,s}(E) = 0 \} .$$

Definition

Let X be a separable metric space. Let $x \in X$,

$$\mathrm{cdim}_g^f(x) = \inf \left\{ s \,\middle|\, \exists^\infty r \; \mathrm{K}_r^f(x) \leq -\log(g(s, 2^{-r})) \right\} .$$

Theorem

Let $E \subseteq X$. Then

$$\dim_{\mathrm{H}}^{(g)}(E) = \min_{B \subseteq \{0,1\}^*} \mathrm{cdim}_g^{f,B}(E).$$

References

- Jack H. Lutz and Neil Lutz, Algorithmic information, plane Kakeya sets, and conditional dimension, STACS 2017
- Elvira Mayordomo, A point-to-set principle for separable metric spaces, in preparation