Martingales and Restricted Ratio Betting

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Motivation

Definition

A *martingale* $d : \Sigma^* \to [0, \infty)$ is a function such that

$$d(\lambda) = 1$$
$$d(w) = d(w0) + d(w1) \quad \forall w \in \Sigma^*$$

A martingale d *succeeds* on $\omega \in \Sigma^\infty$, written $\omega \in S^\infty[d]$, if

$$\limsup_{n\to\infty} d(\omega[0 \ldots n-1]) = \infty.$$

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1. restricting the range to rationals,
2. restricting *wagers* to integers, to a finite set of values etc. This is a restriction on $d(wb) - d(w)$, $w \in \Sigma^*$, $b \in \Sigma$.

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Restricted wagers are investigated in Bienvenu, Stephan, Teutsch 2012, Teutsch 2014, and in Bavly, Peretz 2015.

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$$d(wb) \in \{d(w),\ (1+q)d(w),\ (1-q)d(w)\}.$$

A sequence $\omega \in \Sigma^\infty$ is *simply random* if and only if no simple computable martingale succeeds on it.

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(Masulkar, Nandakumar, Ng)

If we forbid even bets, then the set of random sequences depends on the allowed ratios of betting.

Question

Let A be a finite set of computable numbers in $(0,1)$. An *A-martingale* is a martingale whose ratios of bets $d(wb)/d(w)$, $w \in \Sigma^*$, $b \in \Sigma$, are values in A or $2 - A$.

Note that $1 \notin A$.

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We say that a finite set of ratios B dominates a finite set of ratios A if for every sequence ω that some A-martingale succeeds on, there is some B martingale which succeeds on ω.

Theorem (Masulkar, Nandakumar, Ng)

A finite set of ratios B dominates another finite set of ratios A if and only if $\max A < \max B$.

Single ratio vs. Single ratio

Lemma

Let $A = \{a\}$ *and* $B = \{b\}$*, with* $0 < a < b < 1$*. Then* $S^\infty[A] \subseteq S^\infty[B]$*.*

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Suppose d is an A-martingale that succeeds on ω. Then the B-martingale d' which "imitates" d succeeds on ω: There is a constant $c > 0$ such that

$$d(\omega[0 \ldots n]) = a^k(2-a)^{n-k} > N,$$

then

$$d'(\omega[0 \ldots n]) = b^k(2-b)^{n-k} > N^c.$$

Strict Domination

Lemma

Suppose $A = \{a\}$ *and* $B = \{b\}$ *with* $0 < a < b < 0.5$*. Then there is an* $\omega \in S^{\infty}[A] - S^{\infty}[B]$*.*

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(Proof Idea) Let N_1, N_2, $\ldots$ be a computable enumeration of the c.e. A-martingales.

At stage s, pick a finite extension ω_s of the current candidate ω_{s-1} which satisfies the following:

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At stage s, pick a finite extension ω_s of the current candidate ω_{s-1} which satisfies the following:

- None of $N_1, N_2, \ldots, N_s$ make more money on the extension than they made on ω_{s-1} anywhere along ω_s.
- The B-martingale which always bets $(2 - b)$ on 1 makes more on ω_s than on ω_{s-1}.

Such extensions exist (counting).

Multiple ratios vs. Single ratio

Suppose $A = \{a_1, a_2\}$ and $B = \{b\}$ with $0 < a_1 < a_2 < b$, and $\omega \in S^\infty[A]$. Then one of the following B-martingales succeeds on ω.

- $$N(\sigma\beta) = \begin{cases} bN(\sigma) & \text{if } M(\sigma\beta) < M(\sigma) \\ (2-b)N(\sigma) & \text{otherwise,} \end{cases}$$

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- $$N_2(\sigma\beta) = \begin{cases} bN_2(\sigma) & \text{if } M(\sigma\beta) = a_2 M(\sigma) \\ (2-b)N_2(\sigma) & \text{otherwise.} \end{cases}$$

Definition

An A-martingale *strongly succeeds* on ω if

$$\liminf_{n\to\infty} d(\omega[0\ldots n-1]) = \infty.$$

The "savings account" trick does not work.

Lemma

Let A be a valid ratio set. Then there is a sequence $\omega \in S^{\infty}[A] - S^{\infty}_{str}[A]$.

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Lemma

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(Proof Idea) At every stage s, find a finite extension where every A-martingale $N_1, \ldots, N_s$ makes less than $1/2$ at some point, and on which N_1 attains s at some point.