Yet another characterization of strong jump traceability

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(Joint work with Michael McInerney)

Computability and Randomness

- Study how computation and computability interacts with various concepts.
- Algorithmic randomness:

  Incompressibility, typicality and unpredictability.
- Lots of results relating randomness and computability
  - Interplay between randomness (stochastic properties, patterns) and computability (information content, coding).
  - Tools of computability are used extensively to understand randomness.
  - *Much less often*: Use of randomness (notions) to understand computability.
  - This last approach is the concern of this talk.

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Examples relating computability and randomness

- An extremely fruitful example of interactions of this kind comes from the so-called *lowness notions*.
- “Lowness" refers loosely to any property of a real that indicates that is close to being computable or trivial.
  - For example, weakness when used as an oracle.
  - A (classically) low set A is when $\Phi_e^A(e)$ has the same complexity as $\Phi_e(e)$.
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Examples relating computability and randomness

- Lowness = exhibiting characteristics resembling $\emptyset$.
- The main example from algorithmic randomness is *K-triviality*.
- A real A is K-trivial if it is opposite of being random (Solovay).
- Downey, Hirschfeldt, Nies, Stephan worked on it using ideas from computability.
- K-trivial reals are robust:
  - Low for K, low for random, low for weak 2-randomness, base for randomness, etc.
  - Notice they are all "lowness" properties connected to randomness.
  - Deep results are obtained using tools and intuition from computability.

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- Lowness = exhibiting characteristics resembling $\emptyset$.
- Another example connected to randomness: traceability.

Definition

A trace for a partial function $f :\subseteq \omega \mapsto \omega$ is a sequence of finite sets $\{T_x\}$ of numbers such that for every x, either $f(x)\uparrow$ or $f(x) \in T_x$.

- Origins in the study of cardinal characteristics of the continuum (Bartoszyński) - slaloms.
- Terwijn and Zambella introduced this in the effective context.
- The trace $\{T_x\}$ should be easier to present than f; the value of $f(x)$ is one of finitely many possibilities.

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Traceability: randomness and computability

- For example if g dominates f, then g provides a trace for f.
- Terwijn and Zambella show that “computable traceability" and “c.e. traceability" are related to lowness for Schnorr randomness.
- Are there other ways to express randomness concepts by discrete combinatorial notions? Thinking of K-triviality.
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Traceability: randomness and computability

Definition (Figueira, Nies and Stephan)

- A real A is jump traceable (JT) if the universal A-partial computable function $\Phi_e^A(e)$ has a c.e. trace $\{T_x\}$ such that $\#T_x < h(x)$ for some computable function h.
- A real A is strongly jump traceable (SJT) if for every computable h, the function $\Phi_e^A(e)$ has a c.e. trace $\{T_x\}$ such that $\#T_x < h(x)$.

- (Miller, Nies) Is K-triviality the same as strong jump traceability?
- What started as a seemingly technical definition turned out to be of great interest on its own.
- SJT yielded many further remarkable connections between computability and randomness.

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SJT: randomness and computability

Theorem (Cholak, Downey, Greenberg)

Each c.e. SJT is K-trivial. The converse fails.

- However, it's not the end...

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The c.e. SJT sets form an ideal in the c.e. degrees.

- In contrast, Bickford and Mills show that $\mathbf{0}' = \boldsymbol{a} \cup \boldsymbol{b}$ for some c.e. JT degrees $\boldsymbol{a}, \boldsymbol{b}$.
- Nevertheless, SJTs are fundamentally similar to K-trivials:

Theorem (Greenberg and Nies)

A c.e. set A is SJT iff it obeys every benign cost function.

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- SJTs are shown to be robust with connections to randomness.

Theorem (Greenberg-Nies, Greenberg-Hirschfeldt-Nies, Kucera-Nies, Greenberg-Turetsky)

The following are equivalent for a c.e. set A:

- *A is SJT.*
- *$A \leq_T$ every superlow random.*
- *$A \leq_T$ every superhigh random.*
- *$A \leq_T$ every ω-c.e. random.*
- *$A \leq_T$ some Demuth random.*
- *A is a base for Demuth_{BLR}.*

General results about SJT

- For a time only c.e. SJTs were understood.

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Every SJT is computed by a c.e. SJT.

- This means that SJT is *inherently enumerable*.

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Applications to degree theory

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- A rare application the other way:

Theorem (Downey, Greenberg)

The pseudojump operator obtained by relativising a non-computable c.e. SJT is a natural example of a strongly nontrivial pseudojump operator that cannot be inverted while avoiding an arbitrary uppercone.

- We want more results like this. For example, what we lack is a characterization of SJT in terms of a property that does not mention measure, randomness or any related notion.

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An unlikely candidate

- The property we will use as a candidate comes from the study of c.e. degrees.

Definition

Let $\mathcal{C}$ be a downwards closed class of degrees. Let $\mathrm{PRESERVE}(\mathcal{C})$ be the class of all degrees $\boldsymbol{a}$ such that $\boldsymbol{a} \cup \boldsymbol{w} \in \mathcal{C}$ for every $\boldsymbol{w} \in \mathcal{C}$.

- $\mathrm{PRESERVE}(\mathcal{C})$ is an ideal and $\mathrm{PRESERVE}(\mathcal{C}) \subseteq \mathcal{C}$.
- Ideals are of interest in understanding the structure of c.e. degrees and definability.

Theorem (Cholak, Groszek, Slaman)

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- What can we say about $\mathtt{PRESERVE}(\mathcal{C})$ for various lowness notions $\mathcal{C}$?
- Note: This is interesting only if $\mathcal{C}$ is not itself already an ideal.

Theorem (N)

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- An unexpected application of "diamond classes":

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  - What are their relationships with SJT?

Theorem (McInerney, N)

For each computable h, there is a c.e. set $A \in \texttt{PRESERVE}$(c.e. superlow deg) such that A is not h-JT.

Corollary

- *SJT is a strict subclass of $\texttt{PRESERVE}$(c.e. superlow deg).*
- *$\texttt{PRESERVE}$(c.e. superlow deg) $\subset$ superlow degrees, but there is no computable h such that $\texttt{PRESERVE}$(c.e. superlow deg) $\subseteq$ h-superlow degrees.*

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- We consider expanding the class $\mathcal{C}$ = all JT degrees. (Note there are uncountably many JT degrees).

Theorem (McInerney, N)

Let A be a c.e. SJT. Then deg$(A) \in \texttt{PRESERVE}($all JT deg$)$

- We turn to non-c.e. degrees. By a rather involved argument, we show:

Theorem (McInerney, N)

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  For a Δ^0_2 degree $\boldsymbol{a}$, the following are equivalent:
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- Open question: Can we remove the c.e. assumption for $\texttt{PRESERVE}(\text{all superlow deg})$? JT is used in a very essential way for the above.
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