Pigeons do not jump high

Ludovic Patey
Joint work with Benoit Monin

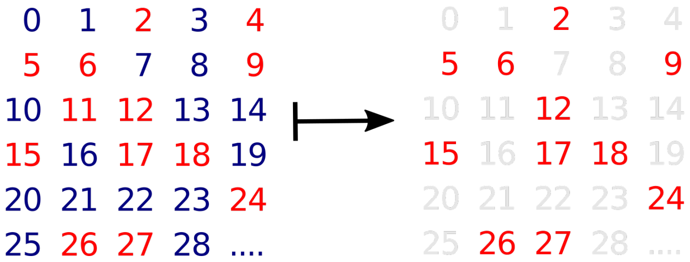
Main results

PIGEONHOLE PRINCIPLE

If you drool infinitely many holes into finitely many pigeons, one pigeon must contain infinitely many holes.

PIGEONHOLE PRINCIPLE

RT^1_k Every k-partition of $\mathbb{N}$ has an infinite subset in one of its parts.



ENCODING SETS WITH THE PIGEONHOLE PRINCIPLE

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Theorem (Dzhafarov, Jockusch)

Suppose $C \not\leq_T \emptyset$. For every set A, there is an infinite subset H of A or $\overline{A}$ such that $C \not\leq_T H$.

ENCODING SETS WITH THE PIGEONHOLE PRINCIPLE

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Suppose $C \not\leq_T \emptyset$. For every set A, there is an infinite subset H of A or $\overline{A}$ such that $C \not\leq_T H$.

Theorem (Monin, P.)

Suppose $C \not\leq_T \emptyset^{(n)}$. For every set A, there is an infinite subset H of A or $\overline{A}$ such that $C \not\leq_T H^{(n)}$ ($n \geq 0$).

Taking $C = \emptyset''$ we obtain solutions of non-high degree.

RESTRICTING TO Δ^0_n INSTANCES

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Theorem (Cholak, Jockusch and Slaman)

For every Δ^0_2 set A, there is an infinite subset H of A or $\overline{A}$ of low_2 degree.

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Theorem (Cholak, Jockusch and Slaman)

For every Δ^0_2 set A, there is an infinite subset H of A or $\overline{A}$ of low$_2$ degree.

Theorem (Monin, P.)

For every Δ^0_n set A, there is an infinite subset H of A or $\overline{A}$ of low$_n$ degree ($n \geq 1$).

By Downey, Hirschfeldt, Lempp and Solomon, we cannot obtain solutions of low$_{n-1}$ degree.

SIDE RESULTS

Theorem (Monin, P.)

Suppose $C \not\in \Sigma^0_n$. For every set A, there is an infinite subset H of A or $\overline{A}$ such that $C \not\in \Sigma^{0,H}_n$ $(n \geq 1)$.

A function f is X-hyperimmune if it is not dominated by any X-computable function.

Theorem (Monin, P.)

Suppose f is $\emptyset^{(n)}$-hyperimmune. For every set A, there is an infinite subset H of A or $\overline{A}$ such that f is $H^{(n)}$-hyperimmune $(n \geq 0)$.

Motivations

REVERSE MATHEMATICS

Foundational program that seeks to determine the optimal axioms of ordinary mathematics.

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Foundational program that seeks to determine the optimal axioms of ordinary mathematics.

$$\mathsf{RCA}_0 \vdash A \leftrightarrow T$$

in a very weak theory RCA_0
capturing computable mathematics

RCA_0

Robinson arithmetics

$$m+1 \neq 0$$
$$m+1 = n+1 \to m = n$$
$$\neg(m < 0)$$
$$m < n+1 \leftrightarrow (m < n \vee m = n)$$

$$m+0 = m$$
$$m+(n+1) = (m+n)+1$$
$$m \times 0 = 0$$
$$m \times (n+1) = (m \times n)+m$$

Σ^0_1 induction scheme

$$\varphi(0) \wedge \forall n(\varphi(n) \Rightarrow \varphi(n+1)) \Rightarrow \forall n \varphi(n)$$

where $\varphi(n)$ is Σ^0_1

Δ^0_1 comprehension scheme

$$\forall n(\varphi(n) \Leftrightarrow \psi(n)) \Rightarrow \exists X \forall n(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is Σ^0_1 with free X, and ψ is Π^0_1.

REVERSE MATHEMATICS

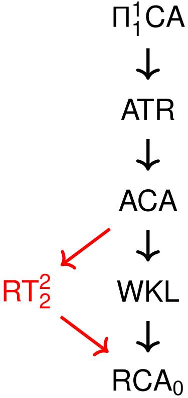
Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

Π^1_1CA
↓
ATR
↓
ACA
↓
WKL
↓
RCA_0

REVERSE MATHEMATICS

Mathematics are computationally very structured



Almost every theorem is empirically equivalent to one among five big subsystems.

Except for Ramsey's theory...

Ramsey's theorem for pairs

And the pigeonhole principle

RAMSEY'S THEOREM

$[X]^n$ is the set of unordered n-tuples of elements of X

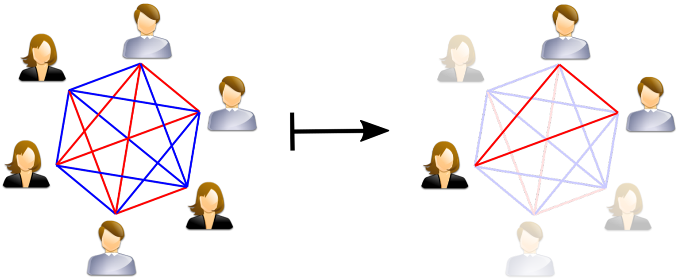
A k-coloring of $[X]^n$ is a map $f : [X]^n \to k$

A set $H \subseteq X$ is homogeneous for f if $|f([X]^n)| = 1$.

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.

RAMSEY'S THEOREM FOR PAIRS

RT^2_k Every k-coloring of the infinite clique admits an infinite monochromatic subclique.



The combinatorial features of RT^n_k reveal the computational features of RT^{n+1}_k

Theorem (Jockusch)

There is a computable instance of RT^3_2 such that every solution computes $\emptyset'$.

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Lemma

Given a function f, there is an f-computable instance of RT^2_2 such that every solution computes a function dominating f.

Define $g : [\omega]^2 \to 2$ by $g(x, y) = 1$ iff $y > f(x)$.

Let $H = \{x_0 < x_1 < \dots\}$ be an infinite g-homogeneous set.

The function $p_H(n) = x_n$ dominates f.

Theorem (Seetapun)

Suppose $C \not\leq_T \emptyset$. For every computable instance of RT^2_2, there is a solution H such that $C \not\leq_T H$.

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Suppose $C \not\leq_T \emptyset$. For every computable instance of RT^2_2, there is a solution H such that $C \not\leq_T H$.

Theorem (Dzhafarov, Jockusch)

Suppose $C \not\leq_T \emptyset$. For every instance of RT^1_2, there is a solution such that $C \not\leq_T H$.

An infinite set C is $\vec{R}$-cohesive for some sets $R_0, R_1, \dots$ if for every i, either $C \subseteq^* R_i$ or $C \subseteq^* \overline{R}_i$.

COH : Every collection of sets has a cohesive set.

Given $f : [\mathbb{N}]^2 \to 2$, define $\langle R_x : x \in \mathbb{N} \rangle$ by $R_x = \{y : f(x, y) = 1\}$.

By COH, there is an $\vec{R}$-cohesive set $C = \{x_0 < x_1 < \dots\}$.

Let $A = \{n \in \omega : \lim_{y \in C} f(x_n, y) = 1\}$.

A is a $\Delta_2^{0,C}$ instance of RT_2^1.

Every RT_2^1-solution to A C-computes an RT_2^2-solution to f.

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{COH} \wedge \Delta^0_2(\mathsf{RT}^1_2)$$

(Cholak, Jocksuch and Slaman)

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{COH} \wedge \Delta^0_2(\mathsf{RT}^1_2)$$

(Cholak, Jocksuch and Slaman)

Theorem (Hirschfeldt, Jocksuch, Kjos-Hanssen, Lempp, and Slaman)

$\mathsf{RCA}_0 \nvdash \mathsf{COH} \rightarrow \Delta^0_2(\mathsf{RT}^1_2)$

Theorem (Chong, Slaman and Yang)

$\mathsf{RCA}_0 \nvdash \Delta^0_2(\mathsf{RT}^1_2) \rightarrow \mathsf{COH}$

Using a non-standard model containing only low sets.

An infinite set C is p-cohesive if it is cohesive for the sequence of primitive recursive functions.

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Theorem (Jockusch and Stephan)

A degree is p-cohesive if and only if its jump is PA over $\emptyset'$.

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- Is there a (Δ^0_2) set A, such that the jump of every infinite subset of A or $\overline{A}$ is of PA degree over $\emptyset'$?

An infinite set C is p-cohesive if it is cohesive for the sequence of primitive recursive functions.

Theorem (Jockusch and Stephan)

A degree is p-cohesive if and only if its jump is PA over $\emptyset'$.

- Is there a (Δ^0_2) set A, such that the jump of every infinite subset of A or $\overline{A}$ is of PA degree over $\emptyset'$?
- Is there a set A, such that every infinite subset of A or $\overline{A}$ is high?

The Ramsey-type hierarchies

RAMSEY-TYPE HIERARCHIES

Ramsey's theorem

Definition

Given a coloring $f : [\mathbb{N}]^n \to k$, a set H is *f-homogeneous* if there exists a color $i < k$ such that $f([H]^n) = i$.

RT^n_k : Every coloring $f : [\mathbb{N}]^n \to k$ has an infinite f-homogeneous set.

Rainbow Ramsey theorem

Definition

A coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ is *k-bounded* if each color is used at most k times. A set H is an *f-rainbow* if f is injective on $[H]^n$.

RRT^n_k : Every k-bounded coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ has an infinite f-rainbow.

RAMSEY-TYPE HIERARCHIES

Thin set theorem

Definition

Given a coloring $f : [\mathbb{N}]^n \to \mathbb{N}$, a set H is *f-thin* if $f([H]^n)$ avoids i.

TS^n : Every coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ has an infinite f-thin set.

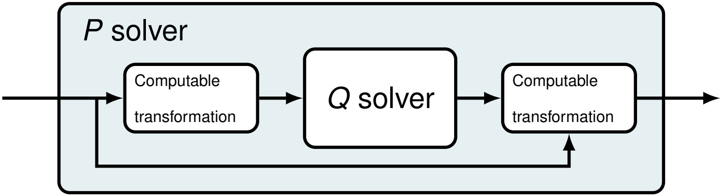
Free set theorem

Definition

Given a coloring $f : [\mathbb{N}]^n \to \mathbb{N}$, a set H is *f-free* if for every $\sigma \in [H]^n$, $f(\sigma) \in H \to f(\sigma) \in \sigma$.

FS^n : Every coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ has an infinite f-free set.

COMPUTABLE REDUCTION



$$\mathrm{P} \leq_c \mathrm{Q}$$

Every P-instance I computes a Q-instance J such that for every solution X to J, $X \oplus I$ computes a solution to I.

The hierarchies over computable reducibility

Definition (Jockusch's bounds)

For every $n \geq 2$,

(i) Every computable P^n-instance has a Π^0_n solution.

(ii) There is a computable P^n-instance with no Σ^0_n solution.

If a hierarchy satisfies Jockusch's bounds, then it is strict over computable reducibility.

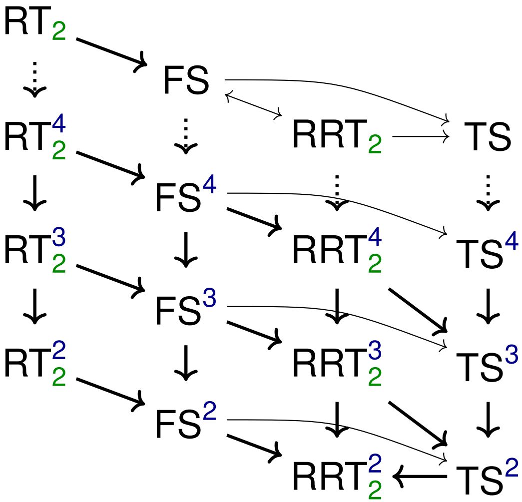
THE HIERARCHIES OVER COMPUTABLE REDUCIBILITY

Theorem

The following satisfy Jockusch's bounds.

- Ramsey's theorem (Jockusch)
- The rainbow Ramsey theorem (Csima, Mileti)
- The free set theorem (Cholak, Giusto, Hirst, Jockusch)
- The thin set theorem (Cholak, Giusto, Hirst, Jockusch)

THE HIERARCHIES OVER COMPUTABLE REDUCIBILITY



RAMSEY-TYPE HIERARCHIES

What about reverse mathematics?

RAMSEY'S THEOREM IN REVERSE MATHS

Theorem (Jockusch)

For every $n \geq 3$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_2 \leftrightarrow \mathsf{ACA}$.

Theorem (Seetapun)

$\mathsf{RCA}_0 \wedge \mathsf{RT}^2_2 \nvdash \mathsf{ACA}$

$$\mathsf{RT}^k_2,\ n \geq 3$$
$$\downarrow$$
$$\mathsf{RT}^2_2$$

RAMSEY-TYPE HIERARCHIES IN REVERSE MATHS

Theorem (Wang)

None of FS, RRT_2 and TS imply ACA over RCA_0.

Theorem (Cholak, Jockusch, Slaman)

Every computable RT^2_2-instance admits a low_2 solution. The same holds for FS^2, RRT^2_2 and TS^2.

Theorem (Wang)

Every computable RRT^3_2-instance admits a low_3 solution.

Ramsey-type hierarchies in reverse maths

Definition (Strong Jockusch's bounds)

For every $n \geq 2$,

(i) Every computable P^n-instance has a low$_n$ solution.

(ii) There is a computable P^n-instance with no Σ^0_n solution.

If a hierarchy satisfies strong Jockusch's bounds, then it is strict over reverse mathematics.

Ramsey-type hierarchies in reverse maths

Do FS, RRT_2 or TS satisfy strong Jockusch's bounds?

RAMSEY-TYPE HIERARCHIES IN REVERSE MATHS

Do FS, RRT_2 or TS satisfy
strong Jockusch's bounds?

Theorem (Monin, P.)

For every Δ^0_n set A, there is an infinite subset H of A or $\overline{A}$ of low$_n$ degree ($n \geq 1$).

Proof techniques

The forcing question

Notion of forcing

$$(\mathbb{P}, \leq)$$

Denotation

$$[c] \subseteq 2^\omega$$

such that $d \leq c \to [d] \subseteq [c]$.

Generic set

$$[\mathcal{F}] = \bigcap_{c \in \mathcal{F}} [c]$$

where $\mathcal{F}$ is a filter.

Forcing relation

$$c \Vdash \varphi(G)$$

where $c \in \mathbb{P}$

Lemma

Let $\mathcal{F}$ be a sufficiently generic filter and $G \in [\mathcal{F}]$ and $\varphi(G)$ be an arithmetical formula. Then

$$\varphi(G) \text{ holds iff } c \Vdash \varphi(G) \text{ for some } c \in \mathcal{F}.$$

Forcing question

$$c \mathrel{?}\!\vdash \varphi(G)$$

where $c \in \mathbb{P}$ and $\varphi(G)$ is Σ^0_n

Lemma

Let $c \in \mathbb{P}$ and $\varphi(G)$ be a Σ^0_n formula.

(a) If $c \mathrel{?}\!\vdash \varphi(G)$, then $d \Vdash \varphi(G)$ for some $d \leq c$;

(b) If $c \mathrel{?}\!\nvdash \varphi(G)$, then $d \Vdash \neg\varphi(G)$ for some $d \leq c$.

EXAMPLE 1: COHEN FORCING

Notion of forcing

$$(2^{<\omega}, \succeq)$$

Denotation

$$[\sigma] = \{G \in 2^\omega : \sigma \prec G\}$$

where $\sigma \in 2^{<\omega}$.

Forcing relation

Δ^0_0	$\sigma \Vdash \varphi(G)$	if $\varphi(\sigma)$ holds
Σ^0_n	$\sigma \Vdash (\exists x)\varphi(G, x)$	if $(\exists w \in \omega)\sigma \vdash \varphi(G, w)$
Π^0_n	$\sigma \Vdash (\forall x)\varphi(G, x)$	if $(\forall w \in \omega)(\forall \tau \succeq \sigma)\tau \vdash \varphi(G, w)$

Forcing question

$$\sigma \mathrel{?{\vdash}} \varphi(G) \text{ if } (\exists \tau \succeq \sigma)\tau \Vdash \varphi(G)$$

EXAMPLE 2: TREE FORCING

A binary tree is a set $T \subseteq 2^{<\omega}$ closed downward under the prefix relation.

Notion of forcing

$$(\mathbb{T}, \subseteq)$$

where $\mathbb{T}$ is the set of infinite computable binary trees

Denotation

$G \in [T]$ if $(\forall \sigma \prec G)\sigma \in T$

where $T \in \mathbb{T}$.

Example 2: Tree forcing

Forcing relation

Σ^0_1 $T \Vdash (\exists x)\varphi(G,x)$ if $(\exists t \in \omega)(\forall \sigma \in T^{[t]})(\exists w \in \omega)\ \varphi(\sigma, x)$

Π^0_1 $T \Vdash (\forall x)\varphi(G,x)$ if $(\forall \sigma \in T)(\forall w < |\sigma|)\ \varphi(\sigma, w)$

Σ^0_n $T \Vdash (\exists x)\varphi(G,x)$ if $(\exists w \in \omega) T \vdash \varphi(G,w)$

Π^0_n $T \Vdash (\forall x)\varphi(G,x)$ if $(\forall w \in \omega)(\forall S \leq T) S \vdash \varphi(G,w)$

Forcing question

Σ^0_1 $T \,?\!\vdash \varphi(G)$ if $T \Vdash \varphi(G)$

Σ^0_n $T \,?\!\vdash \varphi(G)$ if $(\exists S \leq T) S \Vdash \varphi(G)$

APPLICATION 1

Suppose $c\,?\!\vdash \varphi(G)$ is uniformly Σ^0_n whenever $\varphi(G)$ is Σ^0_n

Lemma (Wang)

For every non-Σ^0_n set C, and every Σ^0_n formula $\varphi(G, x)$, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{c \in \mathbb{P} : (\exists w \notin C) c \Vdash \varphi(G, w) \vee (\exists w \in C) c \Vdash \neg\varphi(G, w)\}$$

If C is not Σ^0_n, then it is not $\Sigma^{0,G}_n$ for every sufficiently generic G.

Application 1

Lemma (Wang)

For every non-Σ^0_n set C, and every Σ^0_n formula $\varphi(G, x)$, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{c \in \mathbb{P} : (\exists w \notin C) c \Vdash \varphi(G, w) \vee (\exists w \in C) c \Vdash \neg\varphi(G, w)\}$$

Given $c \in \mathbb{P}$ and $\varphi(G, x)$, define $S = \{w \in \omega : c \,?\!\vdash \varphi(G, w)\}$

S is Σ^0_n but C is not Σ^0_n. Let $w \in S \Delta C$.

- If $w \in S \setminus C$, $d \Vdash \varphi(G, w)$ for some $d \leq c$
- If $w \in C \setminus S$, $d \Vdash \neg\varphi(G, w)$ for some $d \leq c$

APPLICATION 2

Definition

A forcing question $?\vdash$ is compact if for every $c \in \mathbb{P}$ and every formula $\psi(G, x)$, $c \mathrel{?\vdash} (\exists x)\psi(G, x)$ if and only if there is a finite set U such that $c \mathrel{?\vdash} (\exists x \in U)\psi(G, x)$.

A function f is X-hyperimmune if it is not dominated by any X-computable function.

APPLICATION 2

Suppose $c \,?\!\vdash \varphi(G)$ is uniformly Σ^0_n whenever $\varphi(G)$ is Σ^0_n and is compact

Lemma

For every n, every $\emptyset^{(n)}$-hyperimmune function f and every Turing functional Φ_e, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{c \in \mathbb{P} : (\exists w) c \Vdash \Phi_e^{G^{(n)}}(w) \uparrow \vee (\exists w) c \Vdash \Phi_e^{G^{(n)}}(w) < f(w)\}$$

If f is $\emptyset^{(n)}$-hyperimmune, then it is $G^{(n)}$-hyperimmune for every sufficiently generic G.

APPLICATION 2

Lemma

For every n, every $\emptyset^{(n)}$-hyperimmune function f and every Turing functional Φ_e, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{c \in \mathbb{P} : (\exists w) c \Vdash \Phi_e^{G^{(n)}}(w) \uparrow \vee (\exists w) c \Vdash \Phi_e^{G^{(n)}}(w) < f(w)\}$$

Given $c \in \mathbb{P}$ and Φ_e, let $g(w)$ search for a finite set U such that $c \mathrel{?{\vdash}} (\exists x \in U)\Phi_e^{G^{(n)}}(w){\downarrow} = x$ and output $\max U$. g is $\emptyset^{(n)}$-p.c.

- If g is total, then $g(w) < f(w)$ for some w. Then $d \Vdash (\exists x < f(w))\Phi_e^{G^{(n)}}(w){\downarrow} = x$ for some $d \leq c$
- If g is partial, then $g(w) \uparrow$ for some w. Then $d \Vdash (\exists x)\Phi_e^{G^{(n)}}(w) \uparrow$ for some $d \leq c$.

Forcing for RT^1_2

A forcing question for Σ^0_1 formulas.

(Cholak, Jockush and Slaman)

NOTION OF FORCING

$$(F_0, F_1, X)$$

Initial segment (F_0) — Reservoir (X)

- F_i is finite, X is infinite, $\max F_i < \min X$ (Mathias condition)
- $X \in \mathcal{M} \models \text{WKL}$ (Weakness property)
- $F_i \subseteq A_i$ (Combinatorics)

NOTION OF FORCING

Extension

$(E_0, E_1, Y) \leq (F_0, F_1, X)$

- $F_i \subseteq E_i$
- $Y \subseteq X$
- $E_i \setminus F_i \subseteq X$

Denotation

$G_i \in [F_0, F_1, X]_i$

- $F_i \subseteq G_i$
- $G_i \setminus F_i \subseteq X$

$$[E_0, E_1, Y]_i \subseteq [F_0, F_1, X]_i$$

Forcing relation

$$(F_0, F_1, X) \Vdash^i \varphi(G_i)$$

Condition   Formula

Σ^0_1 $(F_0, F_1, X) \Vdash^i (\exists x)\varphi(G_i, x)$ if $(\exists w \in \omega)\varphi(F_i, w)$

Π^0_1 $(F_0, F_1, X) \Vdash^i (\forall x)\varphi(G_i, x)$ if $(\forall E \subseteq X)(\forall w)\varphi(F_i \cup E, w)$

FORCING QUESTION

$$(F_0, F_1, X) \mathrel{?\!\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$$

Lemma

Let $c \in \mathbb{P}$ and $\varphi_0(G), \varphi_1(G)$ be a Σ^0_1 formulas.

(a) If $c \mathrel{?\!\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$, then $d \Vdash^i \varphi_i(G_i)$

(b) If $c \mathrel{?\!\nvdash} \varphi_0(G_0) \vee \varphi_1(G_1)$, then $d \Vdash^i \neg\varphi_i(G_i)$

for some $d \leq c$ and $i < 2$.

FORCING QUESTION

$$(F_0, F_1, X) \mathbin{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$$

is the formula

$$(\forall B_0 \sqcup B_1 = \mathbb{N})(\exists i < 2)(\exists E \subseteq X \cap B_i)\varphi_i(F_i \cup E)$$

or equivalently

$$(\exists H \subseteq_{fin} X)(\forall B_0 \sqcup B_1 = H)(\exists i < 2)(\exists E \subseteq B_i)\varphi_i(F_i \cup E)$$

The formula is $\Sigma_1^{0,X}$

Case 1: $\psi(x, n)$ holds

Letting $B_i = A_i$, there is an extension $d \leq c$ such that

$$d \Vdash^0 \varphi_1(G_0) \quad \text{or} \quad d \Vdash^1 \varphi_1(G_1)$$

Case 2: $\psi(x, n)$ does not hold

The class $\mathcal{C}$ of all $B_0 \sqcup B_1 = \mathbb{N}$ such that

$$(\forall i < 2)(\forall E_i \subseteq X \cap B_i)\Phi_{e_i}^{F_i \cup E_i}(x) \neq n$$

is a non-empty $\Pi_1^{0,X}$ class. Pick $B_0 \sqcup B_1 \in \mathcal{C} \cap \mathcal{M}$.

$$(F_0, F_1, X \cap B_i) \Vdash^i \neg\varphi_i(G_i)$$

Is there a set A, such that the jump of every infinite subset of A or $\overline{A}$ is of PA degree over $\emptyset'$?

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