Effective Classification and Measure for Countable Structures

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(Partially joint work with Johanna Franklin,
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- We will put a measure on the space of (isomorphism types of) algebraic field extensions of $\mathbb{Q}$.

The space of isomorphism types

We work in the language of fields, extended by d-ary *root predicates*:

$$R_d(a_0, \ldots, a_{d-1}) \iff (\exists x)\, x^d + a_{d-1}x^{d-1} + \cdots + a_0 = 0.$$

Let Alg^* be the class of all atomic diagrams of algebraic field extensions of $\mathbb{Q}$, with domain ω. By Gödel coding, we view these diagrams as elements of 2^ω, yielding the quotient topological space $Alg^*/\cong$.

Theorem

There is a computable homeomorphism H from $Alg^*/\cong$ onto 2^ω.

That is, both H and H^{-1} are computed by Turing functionals: $\Phi^{\Delta(F^*)} = H([F^*])$, and $\Psi^J = H^{-1}(J) \in Alg^*$ for all indices $J \in 2^\omega$, with

$$\Psi^{\left(\Phi^{\Delta(F^*)}\right)} \cong F^* \qquad \& \qquad \Phi^{(\Psi^J)} = J.$$

Computing this homeomorphism

$X^4 - 2$ $\qquad$ $\mathbb{Q}(\sqrt[4]{2})$ $\qquad\qquad\qquad$ $\mathbb{Q}$

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Computing this homeomorphism

$X^2 - 2$

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$$\begin{tikzcd}[column sep=small]
\mathbb{Q}(\sqrt[4]{2}) \arrow[dr, no head] & & \mathbb{Q} \arrow[dl, no head] \\
& \mathbb{Q} &
\end{tikzcd}$$

Computing this homeomorphism

$$
\begin{tikzcd}[arrows=dash]
X^2-2 & \mathbb{Q}(\sqrt[4]{2}) \arrow[d] & & \mathbb{Q}(\sqrt{2}) \arrow[dr] & & \mathbb{Q} \arrow[dl] \\
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\end{tikzcd}
$$

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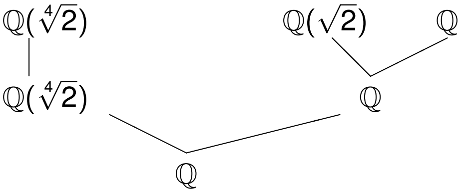
$X^4 - 2$



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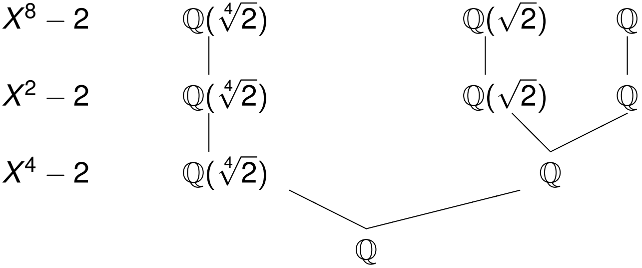

Computing this homeomorphism

$X^2 - \sqrt[4]{2}$

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$X^2 - 2$

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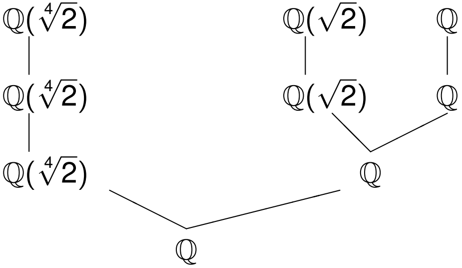
$\uparrow$

List $\overline{\mathbb{Q}}[X]$



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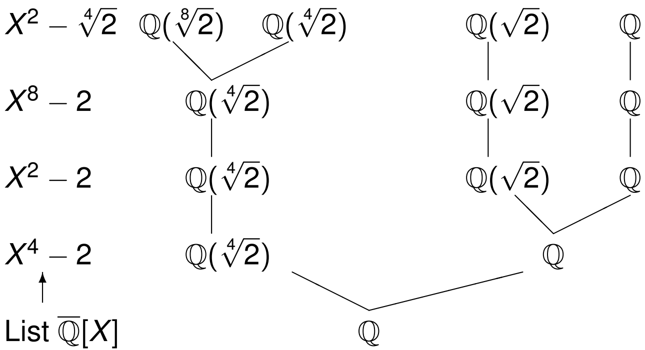

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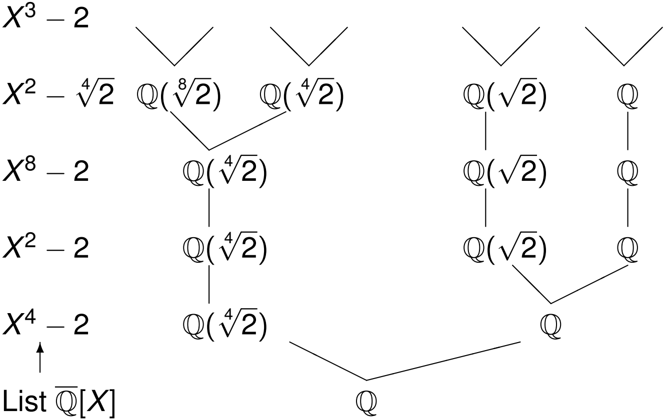

By a result of Kronecker, all finite algebraic extensions $\mathbb{Q}(a_1, \ldots, a_n)$ have splitting algorithms, uniformly in $\vec{a}$. So this tree is computably homeomorphic to 2^ω.

Why the root predicates?

One can consider the space $Alg/\cong$, without the root predicates. This topological space is not homeomorphic to any standard Polish space: it is a *spectral space*, with one element that lies in every non-empty open set, and another element lying in no proper open subset of the space. So it is not useful for classification or measure.

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The space $Alg'/\cong$ consists of the atomic diagrams of jumps of algebraic fields, modulo $\cong$. Here $[\mathbb{Q}']$ forms a singleton open set, being the only field in which this Σ_1^c predicate fails:

$$(\exists p \in \mathbb{Q}[X])(\exists x \in F)\ [p \text{ irreducible of degree } > 1\ \&\ p(x) = 0].$$

The root predicates constitute just enough information for a classification by 2^ω.

A measure on $Alg^*/\cong$

Since $Alg^*/\cong$ is homeomorphic to 2^ω, it seems natural to transfer the Lebesgue measure from 2^ω to $Alg/\cong$. But this requires care. In our first example, the odds of 2 having a square root were $\frac{3}{4}$. In general the measure depends on the ordering $f_0, f_1, f_2, \ldots$ of $\overline{\mathbb{Q}}[X]$.

Fix a computable $\overline{\mathbb{Q}}$, and enumerate $\overline{\mathbb{Q}}[X] = \{f_0, f_1, \ldots\}$. Let $F_\lambda = \mathbb{Q}$. Given $F_\sigma \subset \overline{\mathbb{Q}}$, we find the least i, with f_i irreducible in $F_\sigma[X]$ *of prime degree*, for which it is not yet determined whether f_i has a root in F_σ. Adjoin such a root to $F_{\sigma\hat{}1}$, but not to $F_{\sigma\hat{}0}$. This gives a homeomorphism from 2^ω onto $Alg^*/\cong$, via $h \mapsto \bigcup_n F_{h\restriction n}$.

If we now transfer standard Lebesgue measure to $Alg^*/\cong$, we get a measure in which the odds of 2 having a 1297-th root are $\frac{1}{2}$, but the odds of 2 having a 1296-th root are $\frac{1}{256}$. Is this reasonable?

Haar-compatible measure on $Alg^* / \cong$

A further improvement is to use *Haar-compatible measure* μ on $Alg^* / \cong$. Here the probability of f_σ having a root is deemed to equal $\frac{1}{\deg(f_\sigma)}$. This idea (and the name) are justified by:

Proposition

For every algebraic field F_0 which is normal of finite degree d over $\mathbb{Q}$,

$$\mu(\{[K] \in Alg/\cong : \ F_0 \subseteq K\}) = \frac{1}{d}.$$

Notice that $\frac{1}{d}$ is precisely the measure of the pointwise stabilizer of F_0 within the group $\mathrm{Aut}(\overline{\mathbb{Q}})$, under the Haar measure on this compact group.

Measuring properties of algebraic fields

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In Alg^*, the property of being uniformly computably categorical has measure 1: given two roots x_1, x_2 of the same irreducible polynomial, one can wait for them to distinguish themselves, since with probability 1 there is an f for which $f(x_1, Y)$ has a root in the field but $f(x_2, Y)$ does not. This allows computation of isomorphisms between copies of the field. The process works uniformly except on a measure-0 set of fields.

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Surprisingly, measure-1-many fields in Alg remain uniformly computably categorical even when the root predicates are removed from the language. However, the procedures for computing isomorphisms are not uniform. A single procedure can succeed only for measure-$(1-\epsilon)$-many fields.

Key lemma

Lemma

Let $\alpha \neq \beta \in \overline{\mathbb{Q}}$ be algebraic numbers conjugate over $\mathbb{Q}$. Then, for every finite algebraic field extension $E \supseteq \mathbb{Q}(\alpha, \beta)$, there is a set $D = \{q_0 < q_1 < \cdots\} \subseteq \mathbb{Q}$, decidable uniformly in E, such that for every k, both of the following hold:

$$\sqrt{\alpha + q_k} \notin E(\sqrt{\alpha + q_l}, \sqrt{\beta + q_l} \ : \ l \neq k)(\sqrt{\beta + q_k});$$

$$\sqrt{\beta + q_k} \notin E(\sqrt{\alpha + q_l}, \sqrt{\beta + q_l} \ : \ l \neq k)(\sqrt{\alpha + q_k}).$$

The point is that, for all q in this decidable set D, a field extending E may contain both $\sqrt{\alpha + q}$ and $\sqrt{\beta + q}$, or just one of them, or neither, no matter which other $\sqrt{\alpha + \tilde{q}}$ and/or $\sqrt{\beta + \tilde{q}}$ it contains (for $\tilde{q} \neq q$ in D). The probability of containing $\sqrt{\alpha + q}$ is always $\frac{1}{2}$, independently of all the rest. The proof uses the Hilbert Irreducibility Theorem.

Proving the theorem

Given an $\epsilon > 0$, and a polynomial $f \in \mathbb{Q}[X]$ with roots $\alpha \neq \beta$ in $\overline{\mathbb{Q}}$, fix the set D from the lemma and choose N so large that the odds are $> 1 - \epsilon$ that, in an arbitrary field $\supseteq \mathbb{Q}(\alpha, \beta)$, all of the following hold:

- For at least $0.4N$ of the numbers $q_0, \ldots, q_{N-1}$ in D, $\alpha + q_i$ has a square root in the field.
- For at most $0.35N$ of these numbers, $\alpha + q_i$ and $\beta + q_i$ both have square roots in the field.

The procedure for mapping $\alpha, \beta \in F$ to the right images in a copy $\widetilde{F}$ waits until at least $0.4N$ elements $\sqrt{\alpha + q_i}$ with $i < N$ have appeared in F. Then it maps α to the first $\tilde{\alpha} \in \widetilde{F}$ it finds for which corresponding elements $\sqrt{\tilde{\alpha} + q_i}$ all appear in $\widetilde{F}$.

For polynomials of larger degree, use a similar procedure considering each possible pair of roots of the polynomial.

Randomness and computable categoricity

Theorem (Franklin & M.)

For every Schnorr-random real $h \in 2^\omega$, the corresponding field F_h is uniformly computably categorical, even in the language without the root predicates. However, there exists a Kurtz-random h for which F_h is not u.c.c. (in the language without the root predicates).

This raises a broader idea: we can consider an isomorphism type $[F]$ to be random just if its index in 2^ω is a (ML, Schnorr, etc.)-random set. This appears compatible with Khoussainov's concept of random structures.

In terms of computable structure theory, adding the root predicates ensures that every F^* has an upper cone as its spectrum. But the homeomorphism is stronger than this: one field may be ML-random and another not, even when both have the same upper cone as their spectra.

Finite-branching trees

For another example, consider the class $\mathcal{T}_1$ of all finite-branching infinite trees, under the predecessor function P. As before, we get a topological space $\mathcal{T}_1/\cong$, which is not readily recognizable. The obvious predicates to add are the *branching predicates* B_n:

$$\models_T B_n(x) \iff \exists^{\geq n} y \,(P(y) = x).$$

The enhanced class $\mathcal{T}_1^*$ again has a nice classification. Let $T_{m,0}, T_{m,1}, \ldots$ list all finite trees of height exactly m. Given $T \in \mathcal{T}_1^*$, we can find the unique number $f(0)$ with $T_{1,f(0)} \cong T^{<2}$, where $T^{<2}$ is just T chopped off after level 1.

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Next consider those trees in $T_{2,0}, T_{2,1}, \ldots$ with $T_{2,i}^{<2} \cong T^{<2}$. Choose $f(1)$ so that $T^{<3}$ is isomorphic to the $f(1)$-th tree on this list. Continue choosing $f(2), f(3), \ldots$ recursively this way.

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Measure for FB trees (joint with Franklin)

One can place probability measures on ω^ω by regarding it as the space of all irrational numbers in $(0, 1)$, using continued fractions; or by assigning measure $2^{-(1+\sigma(0))} \cdots 2^{-(1+\sigma(|\sigma|-1))}$ to the set of all paths through σ. These measures are similar, but both have certain problems.

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In all three situations, measure-1-many trees are uniformly c.c. when the branching predicates are included in the language. The interesting question is what happens in the original language.

Computable categoricity for finite-branching trees

- In $\mathcal{T}_0$, uniform computable categoricity has full measure, because every finite tree is u.c.c. However, there is no single procedure Φ which succeeds on measure-0.999-many trees in $\mathcal{T}_0$.

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  one does not know whether to wait for the two left-hand nodes to distinguish themselves: there is positive probability that they will, and also that they will not.
- In $\mathcal{T}_2/\cong$, we conjecture that with the geometric measure, uniform computable categoricity holds on a set of measure 1, but not uniformly. This is somewhat similar to $Alg/\cong$.

(All work here joint with Franklin.)

Does this always happen?

Question

Which classes of countable structures have definable relations which can be Morleyized to make the class (under $\cong$) homeomorphic to 2^ω or ω^ω? And, for standard classes, what are those relations?

- A subring R of $\mathbb{Q}$ is determined by the set of primes p with $\frac{1}{p} \in R$. So the class of all these subrings is computably homeomorphic to 2^ω, in a language with a unary predicate for invertibility.

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- For connected finite-valence graphs, isomorphism is a Σ_3^0 problem. Adding a constant to the language makes it Π_2^0, and adding the valence function makes the class homeomorphic to ω^ω.
- Equivalence structures have Π_4^0 isomorphism problem. Several different possible solutions present themselves here.

In general, one may need to settle for classification as $2^\omega/E$, for certain standard equivalence relations E on 2^ω (or on ω^ω).

TFAb_n: Torsion-free abelian groups of rank n

$\mathrm{TFAb}_n/\cong$ has the indiscrete topology.

TFAb_n^* includes *divisibility predicates* $D_p(x)$ for all primes p. Now, for any n-tuple of independent elements in G, the atomic diagram of G gives us an element of 2^ω, describing which elements of $\mathbb{Q}^n$ (with respect to this basis) lie in G.

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Problem: For different choices of basis, we get different $h \in 2^\omega$. Thus $\mathrm{TFAb}_n^*/\cong$ is computably homeomorphic to $2^\omega/E$, for some ER E.

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For $n > 1$, $E \leq_{\mathbf{0}} E_{\text{set}}$, where $A\ E_{\text{set}}\ B$ means that every column $A^{[n]}$ of A appears as a column of B and vice versa. However, E_{set} is strictly harder than all of these. By results of Hjorth & Thomas, these E must get strictly harder (even in Borel reducibility) as n increases.

RCF_d: Countable archimedean real closed fields

(Joint work with Fokina, Friedman, Rossegger, & San Mauro.)

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Two fields in RCF_d are isomorphic iff they fill the same cuts in $\mathbb{R}$.

Baire space ω^ω is computably homeomorphic to the space of all transcendental $x \in \mathbb{R}$. Let g, h be E-equivalent if they generate the same (strict left) Dedekind cuts. For $d = 1$, we can list out the cuts generated by a given $g \in \omega^\omega$ (as elements of 2^ω). Thus $\mathrm{RCF}_1/\cong$ is classified by ω^ω/E, which computably reduces to E_{set} on 2^ω.

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For $d > 1$, the same procedure works: parse each $g \in \omega^\omega$ into d distinct functions $g^{[i]}$, for $i < d$, and use $g^{[i]}$ to define a real number trancendental over the preceding ones.

Questions: Do these get harder as d increases? And how do they compare with the classes $\mathrm{TFAb}_n^*/\cong$?

Properties of TFAb_n and RCF_d

All these structures (except in RCF_ω) are relatively computably categorical, but for $n, d > 0$, they all require constants. Categoricity questions are less interesting here.

The E-equivalence classes in $2^\omega/E$ (for TFAb_n^*) and ω^ω/E (for RCF_d) all have measure 0. So we do get measures on TFAb_n^* and on RCF_d from those on Cantor and Baire space.

Question: what about randomness? On TFAb_1, the relation E_0 respects all reasonable notions of randomness, so it makes sense to define randomness in TFAb_1. It seems likely that this holds in TFAb_n as well, and in RCF_d for $d < \omega$.