Exponents of irrationality and transcendence and effective Hausdorff dimension

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Abstract

We will discuss the similarities between measuring the describability of a real number in terms of Diophantine Approximation or in terms of Kolmogorov Complexity.

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Irrationality Exponents

Definition (originating with Liouville 1851)

For a real number ξ, the *irrationality exponent of* ξ is the least upper bound of the set of real numbers z such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^z}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

When z is large and $0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^z}$, then p/q is a good approximation to ξ when seen in the scale of $1/q$.

Effective Hausdorff Dimension

Definition (Lutz 2000, Mayordomo 2002)

The *effective Hausdorff dimension* of a real number ξ is the infimum of the numbers z such that for infinitely many ℓ the sequence of the first ℓ digits in the binary expansion of ξ has prefix-free Kolmogorov complexity less than or equal to $z \cdot \ell$.

Example

If ξ has irrationality exponent δ, then ξ has effective Hausdorff dimension less than or equal to $2/\delta$.

An Analogy to Irrationality Exponent

Temporary Definition

For a real number ξ, the *incomputability exponent* of ξ is the least upper bound of the set of real numbers z such that

$$0 < |\xi - R_e| < \frac{1}{e^z}$$

is satisfied by an infinite number of integers e, where R_e is the real number computed by the eth program (for a universal computable enumeration.)

Theorem (Becher, Reimann and Slaman)

For a real number ξ, the effective Hausdorff dimension of ξ is equal to the reciprocal of its incomputability exponent.

Independence

Theorem (Becher, Reimann, and Slaman 2017)

For every $\delta \geq 2$ and every d in $[0, 2/\delta]$, there is a real number ξ such that ξ has irrationality exponent δ and effective Hausdorff dimension d.

There is a Cantor-like set such that, with respect to its uniform measure, almost all real numbers have effective Hausdorff dimension equal to *d* and irrationality exponent equal to δ.

Fourier Dimension

Definition (originating with Salem 1951, (see Mattila, 2015))

The *Fourier dimension* of a set $A \subseteq \mathbb{R}$ is the supremum of the $z \leq 1$ such that there is a measure μ with support A such that for all $t \in \mathbb{R}$, $|\widehat{\mu}(t)| \leq |t|^{-z/2}$.

The Fourier dimension of a set of real numbers is less than or equal to its Hausdorff dimension. (It is also more difficult to evaluate.)

Connection with Normality

Theorem (Kaufman 1981)

For any real number $\delta \geq 2$, the set $\{\xi : \xi$ has irrationality exponent $\delta\}$ has Fourier dimension $2/\delta$, which is also equal to its Hausdorff dimension.

Theorem (based on Davenport, Erdős, and LeVeque 1963, R. Baker)

If $A \subseteq \mathbb{R}$ has strictly-positive Fourier dimension then A has an absolutely normal element.

Further into Diophantine Approximation

The comparison between irrationality exponent and effective Hausdorff dimension has a number-theoretic precedent.

There is a well-developed theory of approximation by algebraic numbers, and there are exponents $\omega_n^*(\xi)$ to measure how well a real number ξ is approximated by algebraic numbers of degree n (see Mahler, 1932a,b; Koksma, 1939; Baker and Schmidt, 1970; Schmidt, 1970; Bugeaud, 2004).

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Definition

For a real number ξ, the $\omega_n^*(\xi)$ is the least upper bound of the set of real numbers z such that

$$0 < |\xi - \alpha| < \frac{1}{H(\alpha)^{(n+1)\cdot z}}$$

is satisfied by an infinite number of algebraic numbers α of degree less than or equal to n and with minimal polynomial of height $H(\alpha)$.

Example

For $\xi \in [0, 1]$, $2 \cdot \omega_1^*(\xi)$ is the irrationality exponent of ξ.

Further into Dimension and Normality

Theorem (Baker and Schmidt 1970)

For any integer n, the set

$$\left\{\xi : |\xi - \alpha| < \frac{1}{H(\alpha)^{(n+1)\cdot\delta}} \text{ for infinitely many algebraic } \alpha \text{ of degree } n\right\}$$

has Hausdorff dimension $1/\delta$.

Further into Dimension and Normality

Theorem (Baker and Schmidt 1970)

For any integer n, the set

$$\left\{\xi : |\xi - \alpha| < \frac{1}{H(\alpha)^{(n+1)\cdot\delta}} \text{ for infinitely many algebraic } \alpha \text{ of degree } n\right\}$$

has Hausdorff dimension $1/\delta$*.*

The Fourier dimension of this set is not known, but we have partial information.

Theorem (Becher, Reimann and Slaman, work in progress)

For sufficiently large δ*, the set*

$$\{\xi : \delta = \omega_n^*(\xi) > \omega_{n-1}^*(\xi)\}$$

has positive Fourier dimension, and so has an absolutely normal element.

Questions

Our reach should exceed our grasp

- Does the Baker-Schmidt Theorem extend to Fourier dimension?
- If $d > 0$ and $\widehat{\mu}(t)$ goes to zero at infinity faster than t^{-d}, what can be said about μ-random reals (beyond absolute normality)?
- What is the exact logical complexity of the set T-numbers, those ξ such that for all n, $\omega_n^*(\xi) < \infty$, and such that $\lim_{n\to\infty} \omega_n^*(\xi) = \infty$?

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Or what's a heaven for?

- Characterize the sequences of the form $(\omega_n^*(\xi) : n \in \mathbb{N})$.
- There are many examples of finiteness theorems in this area for which no computable bounds are known. Are some of these instances of incomputability?

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