Computable planar curves intersect in a computable point

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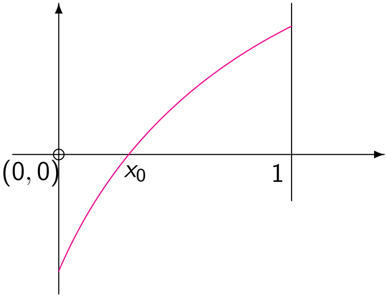
Well-known:

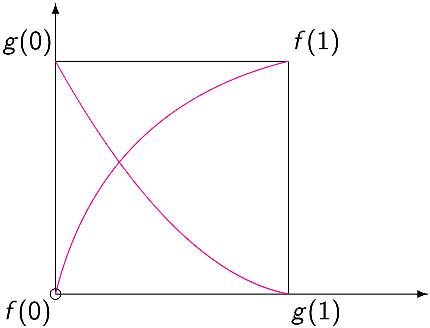
Computable Intermediate Value Theorem

Every computable function $f : [0;1] \to \mathbb{R}$ such that $f(0) < 0$ and $f(1) > 0$ has a computable zero.



The multi-function $f \rightrightarrows x_0$ is not computable.

Suppose $f, g : [0;1] \to [0;1]^2$ and $f(0;1),\ g(0;1) \subseteq (0;1)^2$



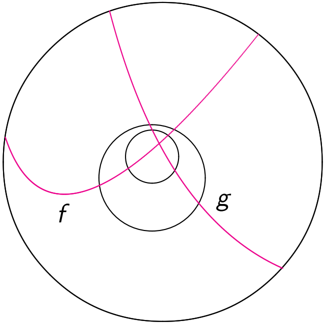
- **Classically:**
  The functions f and g intersect if they are continuous.
- **[Manukyan 1976]**
  There are (Russian-) computable functions f and g which do not intersect.
- Let f, g be (Grzegorczyk-Lacombe-) computable

Theorem [Wei 2017]
The functions f and g intersect in a computable point if they are (Grzegorczyk-Lacombe-) computable.

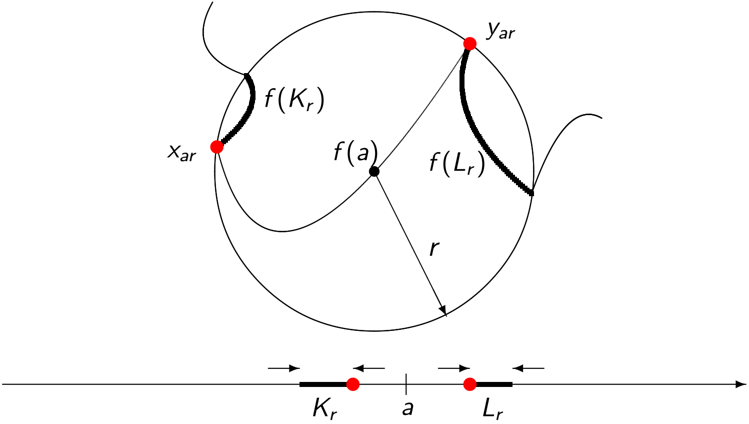
Obvious?

The possibly chaotic behavior of f and g must be controlled.

Proof
Main idea: compute a nested convergent sequence of crossings



The function f entering and leaving a "simple ball" $B(f(a), r)$, $a, r \in \mathbb{Q}$



$\min(L_r) = \sup\{t > a \mid f[a;t] \subseteq B(f(a), r)\}$ comp. from below
$\max(L_r) = \inf\{t > a \mid |f(t) - f(a)| > r\}$ comp. from above
$\min(K_r) = \sup\{t < a \mid |f(t) - f(a)| > r\}$ comp. from below
$\max(K_r) = \inf\{t < a \mid f[t;a] \subseteq B(f(a), r)\}$ comp. from above

For computing the nested sequence of balls we will need "everywhere" balls such that

$$x_{ar}, y_{ar} \notin \mathrm{range}(g)$$

.

$Q \iff$ for all $a, r < s \in \mathbb{Q}, \ldots$

$$(\exists\, t \in \mathbb{Q},\ r \leq t \leq s)\ x_{at}, y_{at} \notin \mathrm{range}(g)$$

(A) If $\neg Q$ then f and g intersect in a computable point.

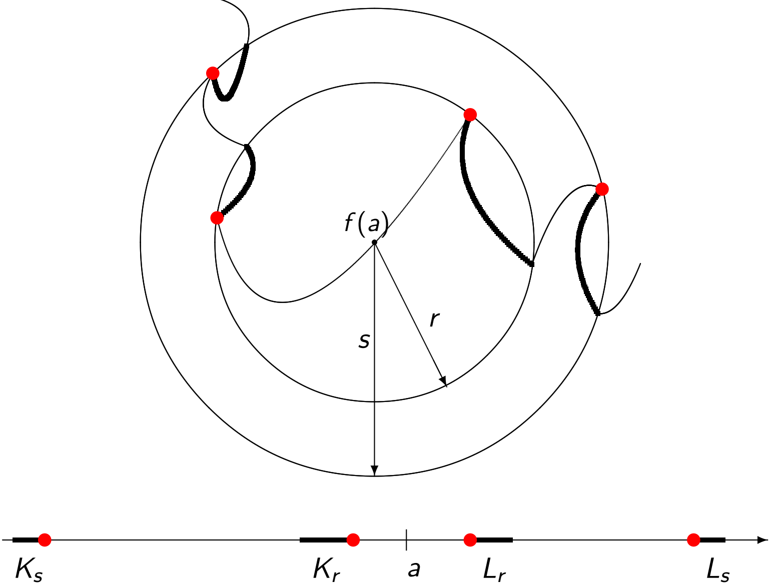
(B) If Q then f and g intersect in a computable point.

Proof for Case (A)

Suppose $\neg Q$: There are $a, r, s \in \mathbb{Q}$, $r < s$ such that

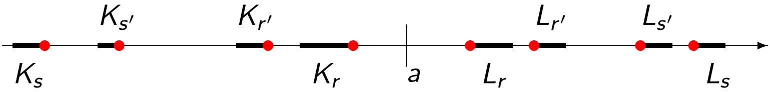
$$(\forall\, t \in \mathbb{Q},\ r \leq t \leq s)\,(x_{at} \in \mathrm{range}(g) \text{ or } y_{at} \in \mathrm{range}(g))$$

x_{at} and y_{at} are in $\mathrm{range}(f)$ but not computable in general.



$$(\forall t \in [r; s]) \left[f \circ \max(K_t) \in \mathrm{range}(g) \text{ or } f \circ \min(L_t) \in \mathrm{range}(g) \right]$$

For $r < r' < s' < s$:



We can compute $r_i, s_i \in \mathbb{Q}$ such that

$$r = r_0 < r_1 < r_2 < \ldots < s_2 < s_1 < s_0 = s$$

and nested sequ. $(I_i)_{i\in\mathbb{N}}$ and $(J_i)_{i\in\mathbb{N}}$ of rat. interv., such that

$$K_{s_i} \cup K_{r_i} \subseteq I_i \,, \qquad L_{r_i} \cup L_{s_i} \subseteq J_i$$

$$\{p\} := \bigcap I_i \,, \qquad \{q\} := \bigcap J_i$$

p and q are computable. By continuity of f,

$$\{f(p)\} := \bigcap f(I_i) \quad \text{and} \quad \{f(q)\} := \bigcap f(J_i)\,.$$

By $\neg Q$.

$$f(I_i) \cap \mathrm{range}(g) \neq \emptyset \ \text{ i.o.} \qquad \text{or} \qquad f(J_i) \cap \mathrm{range}(g) \neq \emptyset \ \text{ i.o.}$$

since $\mathrm{range}(g)$ is compact, hence complete,

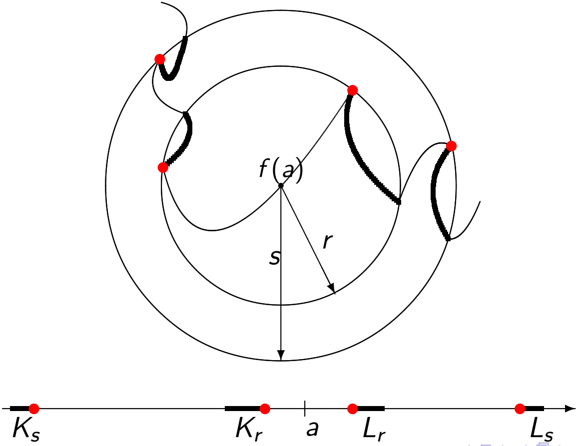
$$f(p) \in \mathrm{range}(g) \qquad \text{or} \qquad f(q) \in \mathrm{range}(g)$$

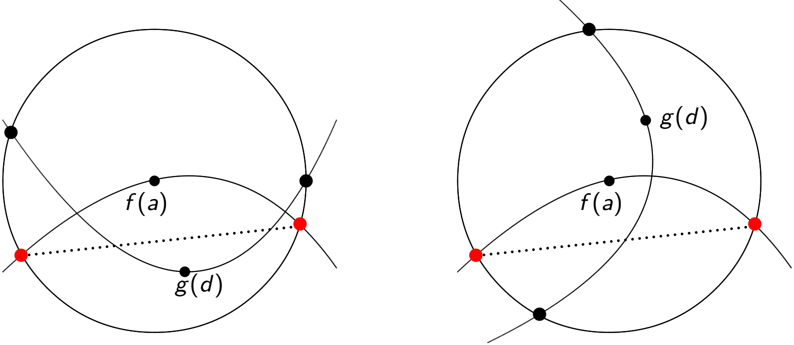
Proof for Case (B)

Suppose Q: For all $a, r, s \in \mathbb{Q}$, $r < s$

$$(\exists\, t \in \mathbb{Q},\ r \le t \le s)\,(x_{at} \notin \mathrm{range}(g) \ \text{ and } \ y_{at} \notin \mathrm{range}(g))$$

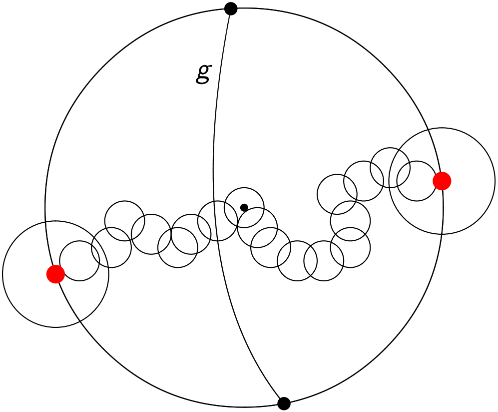
The balls with $x_{at}, y_{rt} \notin \mathrm{range}(g)$ are "dense".





barrier: Ball $B(f(a), r)$ with exit points of f (red) not in $\mathrm{range}(g)$
Case 1: no branch of g crosses the barrier (repellent barrier)
Case 2: some branch of g crosses the barrier (crossing)

Lemma: Every crossing contains a much smaller crossing

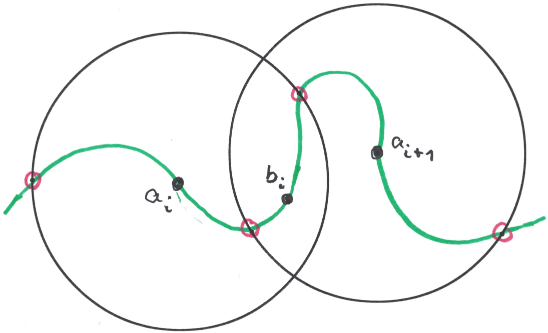


A crossing with balls $B(f(a_i), r_i)$ covering f

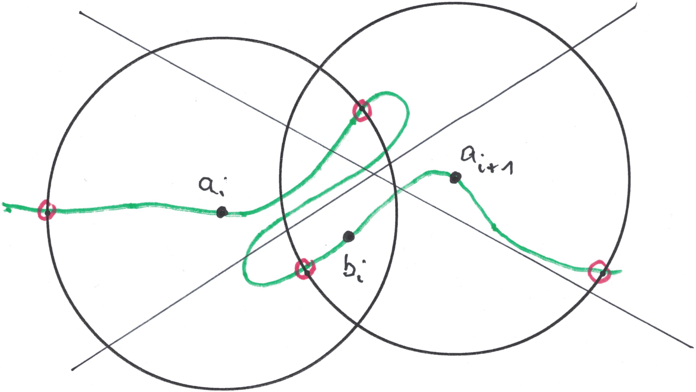
Somewhere g must cross the strip

Times $a_i, b_i \in \mathbb{Q}$ and radii $r_i \in \mathbb{Q}$ can be defined such that f moves through the balls as follows:

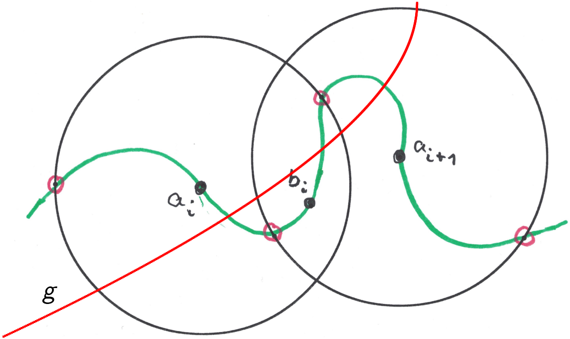
The red points are not in $\mathrm{range}(g)$ (using Condition Q) and ...



but not like this ...



Times $a_i, b_i \in \mathbb{Q}$ and radii $r_i \in \mathbb{Q}$ can be defined such that f moves through the balls as follows: The red points are not in $\mathrm{range}(g)$ and ...



Suppose all barriers $B(f(a_i), r_i)$ are repellent.
g can still cross the strip of balls.

Consider also lens-shaped barriers (intersections of two balls).
g crosses the lens-shaped barrier.
There must be a ball-shaped or a lens-shaped crossing.

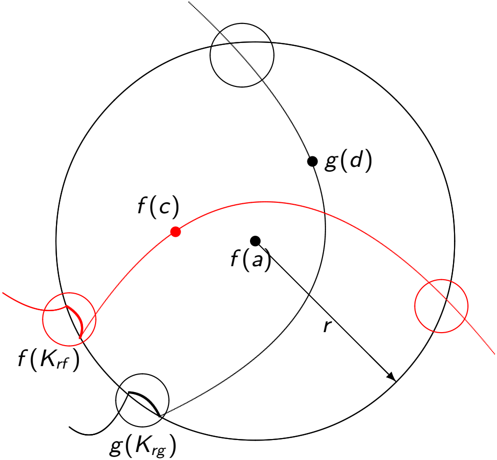
Lemma

Every crossing (unit square, ball-shaped or lens-shaped) contains a much smaller crossing (ball-shaped or lens-shaped).

The set of crossings is not c.e.

Lemma

Every crossing (ball-shaped or lens-shaped) contains a smaller proper crossing (ball-shaped or lens-shaped).



Lemma

The set of proper crossings is c.e.

Theorem [Wei 2017]
The functions f and g intersect in a computable point if they are (Grzegorczyk-Lacombe-) computable.

ArXiv: 2017, Klaus Weihrauch, Computable planar curves intersect in a computable point