Determined Borel codes in Reverse Math

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Borel sets and Borel codes

The *Borel* subsets of a topological space are the smallest collection which

- contains the open sets
- is closed under complements and countable unions.

(and thus countable intersections.)

A *Borel code* is a well-founded tree with

- inner nodes labeled by $\cup$ or $\cap$
- leaves labeled by open sets or their complements.

(note the negations pushed to the leaves.)

Every Borel set can be described by a Borel code.

Borel sets in Reverse Math

Consider Borel subsets of 2^ω in Reverse Math (RM).

Definition (RM): A Borel code is a well-founded tree.

- With conventions to determine $\cup$, $\cap$ and leaf labels.
- If ACA_0 is around, you can assume the tree is labeled.
- The leaves can be labeled with clopen sets $[p]$ with $p \in 2^{<\omega}$, and their complements.

Observe it is easy to get a code for the complement of a set.

- Exchange all $\cup$ and $\cap$.
- Complement all leaves.

Evaluation maps

To determine whether X is in the set coded by T, recursively determine whether X is in the set coded by each subtree.

Definition (RM): If $T \subseteq \omega^{<\omega}$ is a Borel code and $X \in 2^\omega$, an *evaluation map* for X in T is a function $f : T \to \{0, 1\}$ such that

- If σ is a leaf, $f(\sigma) = 1$ if and only if X is in the clopen set associated to σ.
- If σ is a union node, $f(\sigma) = 1$ if and only if $f(\sigma n) = 1$ for some $n \in \omega$.
- If σ is an intersection node, $f(\sigma) = 1$ if and only if $f(\sigma n) = 1$ for all $n \in \omega$.

Evaluation maps can be constructed by *arithmetic transfinite recursion*.

Fact: ATR_0 proves that if T is a Borel code and $X \in 2^\omega$, there is an evaluation map f for X in T.

Borel set membership

Notation: $X \in [T]$ means X is in the set coded by T.
(T is well-founded so there should be no confusion about paths.)

Definition (RM): If T is a Borel code and $X \in 2^\omega$, then $X \in [T]$ if there is an evaluation map f for X in T such that $f(\lambda) = 1$.

$X \in [T]$ is a Σ^1_1 statement.

Fact: ACA_0 proves that if an evaluation map exists, it is unique.

Borel sets "need" ATR_0

Theorem (Dzhafarov, Flood, Solomon, W 2015): The statement "For every Borel code T, there exist X, f such that f is an evaluation map for X in T" implies ATR_0.

If f is an evaluation map for X in T, then $1 - f$ is an evaluation map for X in T's complement.

Corollary: The statement "For every Borel set, either it or its complement is nonempty" is equivalent to ATR_0 over RCA_0.

The property of Baire

Definition: A set $A \subseteq 2^\omega$ has the *property of Baire* if there are open sets U and $\{D_n\}_{n\in\omega}$ such that

- Each D_n is dense.
- For all $X \in \cap_n D_n$, $X \in U \Leftrightarrow X \in A$.

Proposition (FDSW): Over RCA_0, ATR_0 is equivalent to "Every Borel set has the property of Baire."

Proof:
($\Rightarrow$) The standard proof uses arithmetic transfinite recursion.
($\Leftarrow$) If a set has the property of Baire, either it or its complement is nonempty.

($\Leftarrow$) is highly unsatisfactory!

The property of Baire in Reverse Math

How we formalized "Every Borel set has the property of Baire" in RM:

Definition (FDSW): A *Baire code* is open sets $U, V, \{D_n\}_{n\in\omega}$ such that $U \cap V = \emptyset$ and $U \cup V$ and all D_n are dense.

Proposition (FDSW): The following is equivalent to ATR_0 over RCA_0: "For every Borel code T, there is a Baire code $U, V, \{D_n\}$ such that for all $X \in \cap_n D_n$, $X \in U \Rightarrow X \in [T]$ and $X \in V \Rightarrow X \in [T\text{-complement}]$."

Determined Borel codes

Definition: A Borel code T is *determined* if every X has an evaluation map in T. A *determined Borel set* is a Borel set whose code is determined.

This is better:

- In RCA_0, every determined Borel set or its complement is nonempty.
- "Every determined Borel set has the property of Baire" does not trivially imply ATR_0.

Definition: Let DPB be the statement "Every determined Borel set has the property of Baire."

Question: What is the Reverse Math strength of DPB?

Strength of DPB

Proposition: Over RCA_0, DPB implies ACA_0.

Definition (Montalbán 2006): JI (jump iteration) is the statement "For every ordinal α and every X, if $X^{(\beta)}$ exists for all $\beta < \alpha$, then $X^{(\alpha)}$ exists."

Proposition: Over RCA_0, DPB implies JI.

In ω-models, ideals that satisfy jump iteration are HYP ideals.

Some landmarks between ATR_0 and JI

$$
\begin{array}{ccc}
 & & \mathsf{ATR}_0 \\
 & & \Downarrow \\
 & & \Sigma^1_1 - \mathsf{AC} \\
 & & \Downarrow \\
 & & \Delta^1_1 - \mathsf{CA}_0 \\
 & & \Downarrow \\
\mathsf{DPB} & & \text{weak-}\Sigma^1_1 - \mathsf{AC} \\
 & & \downarrow \\
 & & L_{\omega_1,\omega} - \mathsf{CA}_0 \\
 & & \Downarrow \\
 & & \mathsf{JI}
\end{array}
$$

$\mathsf{ATR}_0 \to \mathsf{DPB}$, $\mathsf{DPB} \to L_{\omega_1,\omega} - \mathsf{CA}_0$

Every principle in orange is a *theory of hyperarithmetic analysis*, meaning all its ω-models are HYP ideals, and for every Y, $HYP(Y)$ is a model.

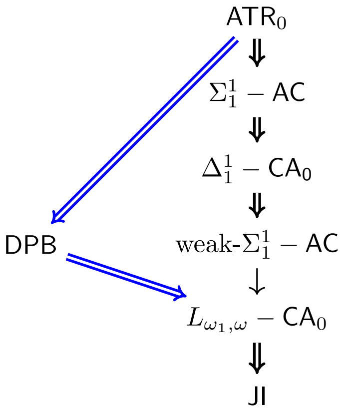
On the location of DPB

Theorem 1: DPB does not hold in HYP.

Theorem 2: There is an ω-model of DPB that does not satisfy ATR_0.

Theorem 3: DPB implies the existence of hyperarithmetic generics in ω-models.
(More formally: If $\mathcal{I}$ is a HYP ideal which satisfies DPB, then for every $X \in \mathcal{I}$, there is a $G \in \mathcal{I}$ such that G is Δ^1_1-generic relative to X.)

Some landmarks between ATR_0 and JI



DPB does not hold in HYP

Theorem 1: DPB does not hold in HYP.

Proof sketch:

- Let U_a denote a canonical universal Σ^0_a set.
- Using overflow, make a computable code for the set

$$\bigcup_a U_a \cap \{X : a \text{ is least s.t. } X \leq H_a\}.$$

- Add "decorations" to the code to give each H_a-computable set an H_a-computable evaluation map.
- It is now a determined Borel code with no HYP Baire code.

DPB does not imply ATR_0

Theorem 2: There is an ω-model of DPB that does not satisfy ATR_0.

Proof sketch:

- Let $G = \bigoplus_i G_i$ be a Δ^1_1 generic with $\omega_1^G = \omega_1$.
- Let $\mathcal{M} = \bigcup_n \Delta^1_1(\bigoplus_{i<n} G_i)$
- Given a determined Borel code, construct a Baire code to agree with the behavior of the G_i which are generic relative to the code.

DPB implies HYP generics exist in ω-models

Theorem 3: DPB implies the existence of hyperarithmetic generics in ω-models.
(More formally: If $\mathcal{M}$ is a HYP ideal which satisfies DPB, then for every $Z \in \mathcal{M}$, there is a $G \in \mathcal{M}$ such that G is Δ^1_1-generic relative to Z.)

Proof sketch:

- If $\mathcal{M}$ has Z but no $\Delta^1_1(Z)$-generics, there is a pseudo-ordinal which $\mathcal{M}$ thinks is well-founded.
- Construct a code for this set by overflow:

$$\bigcup_a U_a \cap \{X : a \text{ is least s.t. } X \text{ is not generic relative to } H_a^Z\}$$

- Decorate the code.
- Every non-$\Delta^1_1(Z)$-generic has an evaluation map, but if the set has the property of Baire, any $X \in \cap_n D_n$ is $\Delta^1_1(Z)$-generic.

Work in progress

Where are the following principles located in the zoo?

- The principle(s) "HYP/Δ^1_1-generics exist" (however one would write that in LSOA).
- Every analytic set has the property of Baire.
- Every determined Borel set is measurable.
- Every determined Borel set has the perfect set property.
- Other theorems involving Borel sets?

References

- Flood, Dzhafarov, Solomon, Westrick. Effectiveness for the Dual Ramsey Theorem. Submitted.
- Montalbán. Indecomposable linear orderings and hyperarithmetic analysis. Journal of Mathematical Logic. 2006.
- Simpson. Subsystems of Second Order Arithmetic.