Universality of the Class of Heyting Algebras

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HKSS-Universality

Hirschfeldt, Khoussainov, Shore, and Slinko (2002) introduced a notion of universality for a class of countable structures.

Here we call this notion **HKSS-universality** (a formal definition is given in the appendix).

Inside an HKSS-universal class, one can realize *any possible*:

- degree spectrum of a structure,
- degree spectrum of a relation,
- $\mathbf{d}$-computable dimension of a computable structure,
- categoricity spectrum of a computable structure.

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The following classes *are not* HKSS-universal:

(a) linear orders;

(b) Boolean algebras.

[Richter 1977, 1981] — for degree spectra;

[Goncharov and Dzgoev 1980; Remmel 1981] — for computable dimensions.

Heyting algebras

A structure $\mathcal{H} = (H; \vee, \wedge, \to, 0, 1)$ is a **Heyting algebra** if $(H; \vee, \wedge, 0, 1)$ is a bounded distributive lattice, and for every $a, b \in H$, the element $a \to b$ is the greatest element in the set $\{x : a \wedge x \leq b\}$.

- Every Boolean algebra can be viewed as a Heyting algebra: $(a \to b) = \neg a \vee b$.
- A linear order with the least and the greatest elements is also a Heyting algebra:

$$a \to b = \begin{cases} 1, & \text{if } a \leq b, \\ b, & \text{if } a > b. \end{cases}$$

Computable Heyting algebras

Heyting algebras can realize some interesting computability-theoretic properties:

- For any Turing degree $\mathbf{d}$, there is a Heyting algebra $\mathcal{H}$ such that $\mathrm{DgSp}(\mathcal{H}) = \{\mathbf{c} : \mathbf{c} \geq \mathbf{d}\}$ [Turlington 2010].
- For a computable successor ordinal $\alpha \geq 3$, computable Heyting algebras realize all possible $\mathbf{0}^{(\alpha)}$-computable dimensions [B. 2017].

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Theorem (B. 2021)

The class of Heyting algebras with distinguished atoms and coatoms is HKSS-universal.

Question 1

Is the class of Heyting algebras HKSS-universal?

One can ask more restricted questions: for example,

Question 1.a

Can computable Heyting algebras realize all possible computable dimensions?

Note that Turlington (2010) proved that a free Heyting algebra has computable dimension either 1 or ω.

Distributive lattices

Hirschfeldt, Khoussainov, Shore, and Slinko (2002) proved that the class of lattices is HKSS-universal.

In general, the lattices constructed in their paper are *not modular*.

Question 2

Is the class of distributive lattices HKSS-universal?

Question 2.a

Can computable distributive lattices realize all possible computable dimensions?

On a related note, there exists a computably categorical distributive lattice, which is not relatively Δ_2^0-categorical [B., Frolov, Kalimullin, and Melnikov 2017].

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Appendix: The formal definition of HKSS-universality

A class of structures K is **HKSS-universal** if for every countable, automorphically nontrivial graph G, there exists a countable, automorphically nontrivial structure $\mathcal{S}_G \in K$ with the following properties:

(1) $\mathrm{DgSp}(\mathcal{S}_G) = \mathrm{DgSp}(G)$.

(2) If G has a computable copy, then:

   a. For every Turing degree $\mathbf{d}$, $\dim_{\mathbf{d}}(\mathcal{S}_G) = \dim_{\mathbf{d}}(G)$.

   b. For any element $c \in G$, there exists $a \in \mathcal{S}_G$ such that $\dim_{\mathbf{0}}(\mathcal{S}_G, a) = \dim_{\mathbf{0}}(G, c)$.

   c. If $R \subseteq \mathrm{dom}(G)$, then there exists a relation $Q \subseteq \mathrm{dom}(\mathcal{S}_G)$ such that $\mathrm{DgSp}_{\mathcal{S}_G}(Q) = \mathrm{DgSp}_G(R)$. In addition, if R is intrinsically c.e., then so is Q.