Open questions on Poisson generic reals

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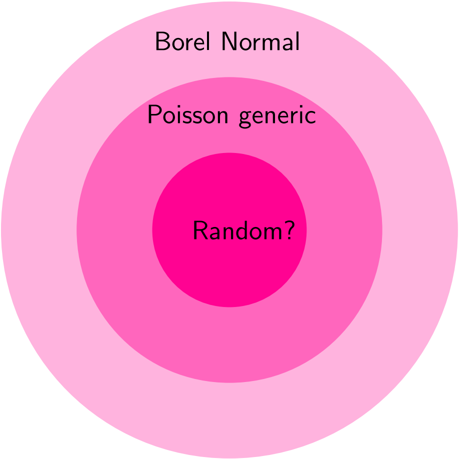
Are all Martin-Löf random reals Poisson generic ?



Towards the definition of Poisson generic reals

Consider the set $\{0, 1\}$.

By uniform probability each word of length n has probability 2^{-n}.
If we toss a coin N times, we expect that each block of length n occurs about $N/2^n$ many times plus minus ϵN.

This gives us an expectation of how many times we will find each length n-word only in case n is smaller than $\log N$, this is only in case 2^n is smaller than N.

Because when $2^n = N$ each block of length n is expected to occur about $N/2^n = 1$ times plus minus ϵN, this is 0, 1, 2 or more times ....
This is not very informative.

Poisson generic reals

Definition (Zeev Rudnick; Peres and Weiss)

A binary sequence x is Poisson generic if for all $\lambda > 0$ and all integers $k \geq 0$,

$$\lim_{n\to\infty} \frac{\#\text{ length-}n\text{ words occur exactly }k\text{ times in first }[\lambda 2^n]\text{ symbols of }x}{\#\text{ length-}n\text{ words}} = e^{-\lambda}\frac{\lambda^k}{k!}.$$

Poisson generic reals

Convergence to the Poisson law. Suppose an event X has probability p. The probability of exactly k occurrences of X in N independent draws is

$$\binom{N}{k} p^k (1-p)^{N-k}$$

Let $\lambda > 0$ and for each N let $p = \lambda/N$. So, for each fixed integer $k \geq 0$,

$$\begin{aligned} \lim_{\substack{N\to\infty \\ p=\lambda/N}} \binom{N}{k} p^k (1-p)^{N-k} &= \lim_{N\to\infty} \binom{N}{k} \left(\frac{\lambda}{N}\right)^k \left(1-\frac{\lambda}{N}\right)^{N-k} \\ &= \lim_{N\to\infty} \frac{N(N-1)\cdots(N-k+1)}{N^k} \left(1-\frac{\lambda}{N}\right)^N \frac{\lambda^k}{k!} \\ &= e^{-\lambda}\frac{\lambda^k}{k!} \end{aligned}$$

and it holds that $\sum_{k\geq 0} e^{-\lambda}\frac{\lambda^k}{k!} = 1$.

Poisson generic reals

Theorem (Peres and Weiss)

Almost all (Lebesgue measure) real numbers are Poisson generic.

Theorem (Peres and Weiss)

Poisson generic reals are Borel normal.

Theorem (Peres and Weiss)

Champernowne sequence is not Poisson generic.

Problem (Peres and Weiss)

Give an example of a Poisson generic real.

Benjamin Weiss. Random-like behavior in deterministic systems
Institute for Advanced Study, Princeton University USA. 16 June 2010.
https://www.youtube.com/watch?v=8AB7591De68&t=1567s

Poisson generic reals

Question 1

Is there a computable Poisson generic real?

Question 2

Are all Martin-Löf random reals Poisson generic?

Question 3

Is it possible to characterize Poisson generic reals with some complexity?

Normal numbers are incompressible by finite-state automata with input and output.

Question 4

Prove that the set of Poisson generic reals is Π^0_3-complete.