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The uniform strength of the open Ramsey theorem

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Joint work with Alberto Marcone

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Open and clopen Ramsey theorems

$[\mathbb{N}]^{\mathbb{N}}$: space of strictly increasing sequences of natural numbers

$[f]^{\mathbb{N}}$: subsequences of $f \in [\mathbb{N}]^{\mathbb{N}}$

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Every open/clopen subset of $[\mathbb{N}]^{\mathbb{N}}$ has a homogeneous solution.

Open-RT as a problem

full version: given an open set, find a homogeneous solution for it (which may either land in it or avoid it).

$$\boldsymbol{\Sigma}^0_1\text{–}\mathsf{RT}$$

weak versions: given an open set with no solutions that avoid it (resp. land in it), find a solution that lands in it (resp. avoids it);

$$\mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1} \quad \mathsf{wFindHS}_{\boldsymbol{\Pi}^0_1}$$

strong versions: given an open set with some solution that lands in it (resp. avoids it), find a solution that lands in it (resp. avoids it);

$$\mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1} \quad \mathsf{FindHS}_{\boldsymbol{\Pi}^0_1}$$

Open-RT in the Weihrauch lattice

$\mathsf{C}_{2^{\mathbb{N}}}$: given an ill-founded tree $T \subset 2^{\mathbb{N}}$, produce a path $x \in [T]$

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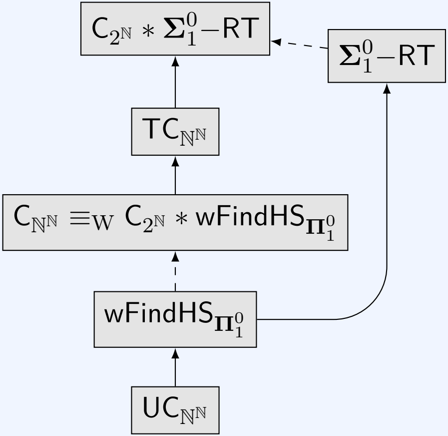
Theorem (Brattka, de Brecht, Pauly), (Kihara, Marcone, Pauly)

$\mathsf{C}_{2^{\mathbb{N}}} <_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^{\mathbb{N}}} <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} <_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^{\mathbb{N}}}$

Open-RT in the Weihrauch lattice

Theorem (Marcone, V.)



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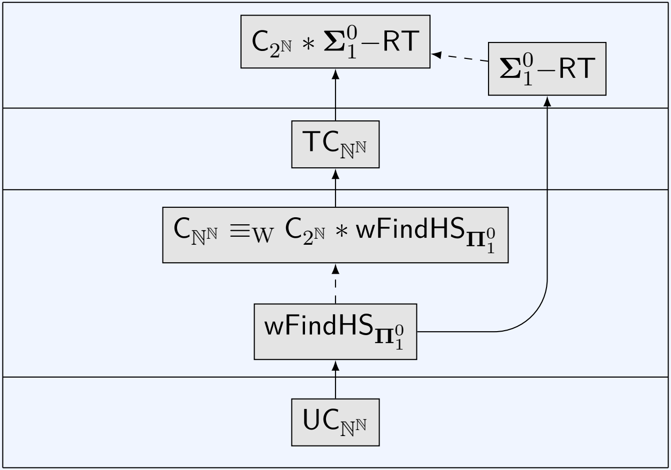

Open questions

1. $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_W \mathbf{\Sigma}^0_1{-}\mathsf{RT}$?
2. $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_W \mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$?

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1. $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_{\mathrm{W}} \boldsymbol{\Sigma}_1^0\mathsf{-RT}$?
2. $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_{\mathrm{W}} \mathsf{wFindHS}_{\boldsymbol{\Pi}_1^0}$?

A. Marcone and M. Valenti, *The open and clopen Ramsey theorems in the Weihrauch lattice*, The Journal of Symbolic Logic (2021?).