Department of Mathematics, Computer Science, Physics
University of Udine

The uniform strength of bad sequences

Manlio Valenti
manliovalenti@gmail.com

Joint work with Jun Le Goh and Arno Pauly

Oberwolfach Apr 30, 2021

Main definitions

"Find a descending sequence through an ill-founded linear order"

$$\mathsf{DS} :\subseteq \mathrm{LO} \rightrightarrows \mathbb{N}^\mathbb{N} := \leq_L \mapsto \{x \in \mathbb{N}^\mathbb{N} : (\forall i)(x(i+1) <_L x(i))\}$$

"Find a bad sequence through a non-well quasi-order"

$$\mathsf{BS} :\subseteq \mathrm{QO} \rightrightarrows \mathbb{N}^\mathbb{N} := \preceq_Q \mapsto \{x \in \mathbb{N}^\mathbb{N} : (\forall i < j)(x(i) \not\preceq_Q x(j))\}$$

Main definitions

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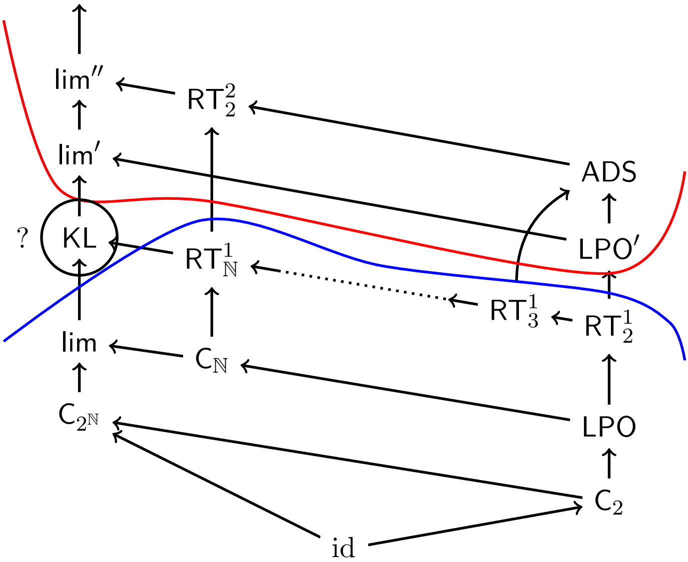
For $\mathbf{\Gamma} \in \{\mathbf{\Sigma}^0_k, \mathbf{\Pi}^0_k, \mathbf{\Delta}^0_k, \mathbf{\Sigma}^1_1, \mathbf{\Pi}^1_1, \mathbf{\Delta}^1_1\}$ we can consider

$$\mathbf{\Gamma}\text{-}\mathsf{DS} :\subseteq \mathbf{\Gamma}(\mathrm{LO}) \rightrightarrows \mathbb{N}^{\mathbb{N}} \qquad \mathbf{\Gamma}\text{-}\mathsf{BS} :\subseteq \mathbf{\Gamma}(\mathrm{QO}) \rightrightarrows \mathbb{N}^{\mathbb{N}}$$

Same as above but answering the question "$a \leq_L b$" (or "$a \preceq_Q b$") is Γ relative to the input.

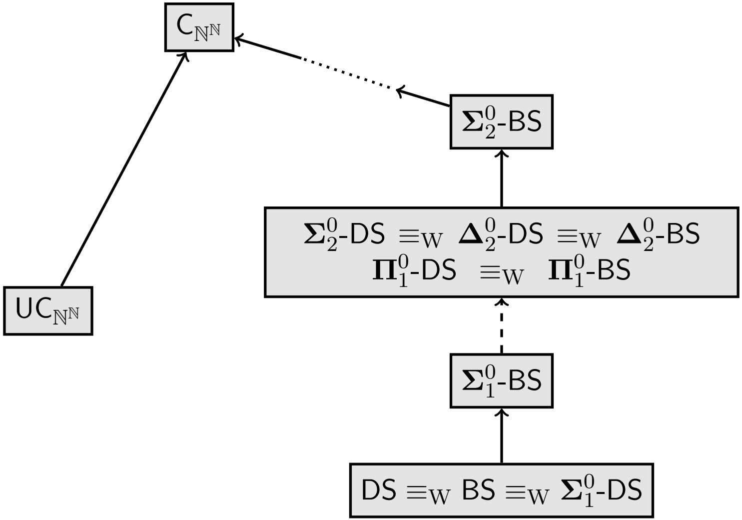
How strong is DS?



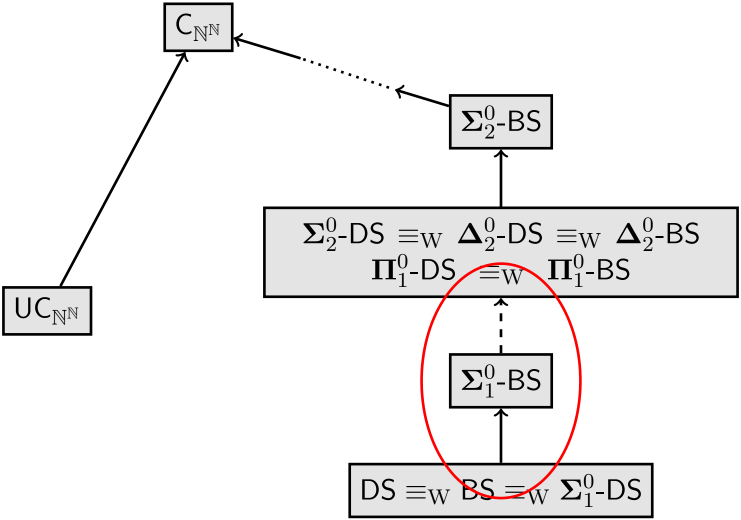
Open questions

1. $\mathsf{KL} \leq_{\mathrm{W}} \mathsf{DS}$?

The arithmetic DS hierarchy



The arithmetic DS hierarchy



More definitions

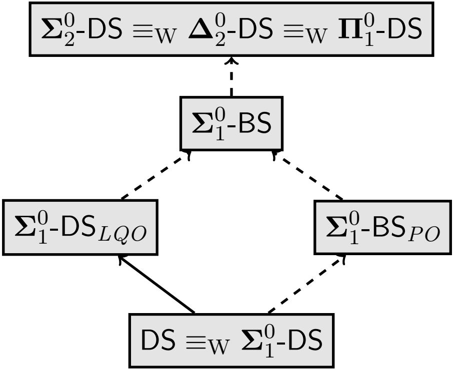
$\mathbf{\Gamma}\text{-}\mathsf{BS}_{LQO}$: Restriction of $\mathbf{\Gamma}\text{-}\mathsf{BS}$ to quasi-order with linearly ordered equivalence classes

$\mathbf{\Gamma}\text{-}\mathsf{BS}_{PO}$: Restriction of $\mathbf{\Gamma}\text{-}\mathsf{BS}$ to partial orders

More definitions

$\mathbf{\Gamma}$-BS_{LQO} : Restriction of $\mathbf{\Gamma}$-BS to quasi-order with linearly ordered equivalence classes

$\mathbf{\Gamma}$-BS_{PO} : Restriction of $\mathbf{\Gamma}$-BS to partial orders



Open questions

1. $\mathsf{KL} \leq_W \mathsf{DS}$?
2. $\mathbf{\Sigma}^0_1\text{-}\mathsf{BS} \leq_W \mathbf{\Sigma}^0_1\text{-}\mathsf{BS}_{LQO}$?
3. $\mathbf{\Delta}^0_2\text{-}\mathsf{DS} \leq_W \mathbf{\Sigma}^0_1\text{-}\mathsf{BS}$?
4. What can be said about $\mathbf{\Sigma}^0_1\text{-}\mathsf{BS}_{PO}$?

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2. $\boldsymbol{\Sigma}^0_1\text{-}\mathsf{BS} \leq_{\mathrm{W}} \boldsymbol{\Sigma}^0_1\text{-}\mathsf{BS}_{LQO}$?
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4. What can be said about $\boldsymbol{\Sigma}^0_1\text{-}\mathsf{BS}_{PO}$?

J. L. Goh, A. Pauly, and M. Valenti, *Finding descending sequences through ill-founded linear orders*, The Journal of Symbolic Logic (2021?), available at https://arxiv.org/abs/2010.03840.