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Computability Theory (hybrid meeting)

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Abstract. Over the last decade computability theory has seen many new and fascinating developments that have linked the subject much closer to other mathematical disciplines inside and outside of logic. This includes, for instance, work on enumeration degrees that has revealed deep and surprising relations to general topology, the work on algorithmic randomness that is closely tied to symbolic dynamics and geometric measure theory. Inside logic there are connections to model theory, set theory, effective descriptive set theory, computable analysis and reverse mathematics. In some of these cases the bridges to seemingly distant mathematical fields have yielded completely new proofs or even solutions of open problems in the respective fields. Thus, over the last decade, computability theory has formed vibrant and beneficial interactions with other mathematical fields.

The goal of this workshop was to bring together researchers representing different aspects of computability theory to discuss recent advances, and to stimulate future work.

Mathematics Subject Classification (2010): 03Dxx; Secondary: 03B30, 03C57, 03Fxx, 68Qxx.

Introduction by the Organizers

Computability theory is one of the main branches of mathematical logic. It explores the computational limitations of mathematics. Classical concepts at the center of the area include the degrees of unsolvability, the arithmetical and analytic hierarchies, and many other methods of calibrating relative complexity. Principal applications have been to algorithmic randomness, mathematical logic, algebra, analysis, and proof theory. A number of deep tools have been developed in the area, including priority methods, effective forcing methods, and sophisticated coding techniques. While there have been several memorable recent results clarifying the pure theory, much of current research is devoted to using these techniques to distill the effective content of applications and give insight into applications.

Here are some of the special topics that we managed to draw together at this meeting in addition to general computability theory:

The Enumeration Degrees in Effective Topology. Computability theory formalizes notions of *relative complexity*: it classifies objects by algorithmic information content, defining when one object contains more information than another. The most commonly used tool is *Turing reducibility*, which measures the complexity of functions $f\colon \mathbb{N} \to \mathbb{N}$. Turing reducibility is extended by *enumeration reducibility*, which allows us to study more objects.

An early example of this phenomenon is given by C. F. Miller (unpublished) and M. Ziegler who applied enumeration reducibility in group theory, to state and prove an extension of Higman's embedding theorem for finitely generated groups. More recently, J. Miller showed that Turing reducibility is not sufficient to measure the complexity of continuous functions on the unit interval, but enumeration reducibility is. In this work he introduced the continuous degrees, a degree structure that sits between the Turing degrees and the enumeration degrees. The proof that the continuous degrees are strictly larger than the Turing degrees invokes nontrivial topological theorems.

Kihara and Pauly use the continuous degrees to solve an open problem in descriptive set theory, following Jayne's study of restricted isomorphisms on topological spaces; they constructed continuum-many spaces, which are pairwise non isomorphic by finite-level Borel isomorphisms.

Definability results show how topological notions are reflected in the abstract structure of enumeration degrees of complexity. Cai, Ganchev, Lempp, Miller and Soskova proved that the Turing degrees have a natural first order definition in the partial order of enumeration degrees, solving an longstanding open problem set by Rogers in the 1960's. Andrews, Igusa, J. Miller, and M. Soskova proved that the continuous enumeration degrees also form a definable class, characterized by its relation to the Turing degrees. Another class of degrees (the *cototal* degrees) has realizations in many parts of mathematics including graph theory, symbolic dynamics by McCarthy, and once again via a structural feature of some enumeration degrees by Miller and Soskova. Kihara, Ng, and Pauly characterized them as the degrees of points in computable G_δ spaces. Thus, the topological considerations again give a good approach for the fine-grained study of the enumeration degrees, as both previously studied substructures as well as new ones of interest to computability theorists appear in this fashion. In the words of the authors: "We have only just started to reveal the topological aspects of the enumeration degrees."

Algorithmic Randomness. Algorithmic randomness attempts to answer questions such as "what does it mean for an individual binary sequence to be random?".

It develops tools to compare different strengths of randomness, and relates them to the Turing and enumeration degrees on one hand, and notions from analysis on the other. Recently, research has focused on effective properties of dynamical systems. Effective symbolic dynamics has been studied, among others, by Simpson, who related complexity, dimension, and entropy. Westrick showed that "sea of squares" subshifts are sofic. This built on celebrated work by Hochman and Meyerovitch who characterized the entropies of higher dimensional shifts of finite type as those which are the reals which are computably approximable from the right. On the measure side, effective ergodic theory has been studied, relating notions of randomness to different ergodic theorems. Yampolsky and co-authors explored the effective properties of complex dynamics.

Mayordomo showed that effective Hausdorff dimension can be characterized by descriptive complexity. A striking recent body of work by N. Lutz and co-authors makes use of this idea to give new results in classical geometric measure theory. For example, Lutz and Stull extended Marstrand's projection theorem, that says that for an analytic set E, for almost all lines ℓ through the origin, the Hausdorff dimension of E's projection onto ℓ is maximal. They also gave improved lower bounds on the dimension of generalized sets of Furstenberg type. J. and N. Lutz used Kolmogorov complexity to give a new proof of Davies' theorem about the dimension of Kakeya sets in the plane (sets that contain unit line segments in every direction).

Interactions Between Set Theory and Computability. There have always been close connections between computability and set theory, harking back to the diagonal method used by both Cantor, Gödel and Turing to show the uncountability of the reals on one hand, and the undecidability of the halting problem on the other. In the late 1950s and early 1960s, work of Spector, Gandy, Kreisel and Sacks has laid the basis for effective descriptive set theory, which blends both areas seamlessly. On a technical level, Sacks and his followers showed how common ideas lie behind the technique of forcing, which is now a fundamental tool in both computability and set theory.

Recent work in effective descriptive set theory has seen unexpected applications and has developed in new directions. Here we can mention, for example, Kihara's use of the Shore-Slaman join theorem on the one hand, very recent work by Day and Marks (in preparation) which has utilized iterated priority arguments to solve the long-standing decomposition problem of functions in the Borel hierarchy. Involving measure, another example is the study of higher randomness, which uses set-theoretic techniques to study randomness at the level of analytic sets. Hjorth, Nies, Chong, and Yu showed that modern notions of algorithmic randomness can behave in surprising ways when their higher analogues are considered. Bienvenu, Greenberg and Monin explored the role of continuity in the theory of randomness. In parallel, set theorists have explored classes of real numbers computable by infinite time Turing machines, and how these classes related to Gödel's constructible hierarchy. This in turn gives rise to new notions of randomness.

Computable Model Theory. Computable model theory investigates the effective content of structures: either in particular algebraic classes (groups, fields, graphs, ...) or in generality. The main theme is understanding how algebraic structure and computable properties affect each other. For example, a central notion is that of *algorithmic dimension*: the number of computable copies up to computable isomorphism. Goncharov used the theory of numberings to show that any finite number is possible. Recently, Fokina, Kalimullin and R. Miller have suggested a more flexible invariant based on the Turing reducibility: the *degree of categoricity*, regarding the complexity of not only the structures but also of the isomorphisms between them. This was extended by Bazhenov, Kalimullin and Yamaleev and Csima and Stephenson. These notions suggest many interesting questions, one of which was very recently solved by Turetsky (unpublished).

Also there are different approaches and questions related to the computability of uncountable objects, as well as the numbering theory is under active development which uses the methods both from Classical Computability Theory and from Computable Model Theory.

Reverse Mathematics and Weihrauch Complexity. Reverse mathematics, developed by Harvey Friedman and Steve Simpson, is a branch of proof theory which attempts to answer the question "what axioms of mathematics are actually necessary for proving a given mathematical theorem?". The context is usually second-order arithmetic, working over a weak base theory. This study has very close connections to computability, as the natural models involved are well-behaved collections of Turing degrees, the set-existence axioms investigated are related to logical definability and computational strength. For example, the common base theory RCA_0 states, roughly, that computable objects exist, and that induction can be performed on computably enumerable sets.

More recently, it turned out that there are certain theorems that do not allow for such a simple classification. Perhaps the best studied such example is Ramsey's theorem for pairs that yields a class on its own and is not linearly linked to the other classes above. A major effort to fully understand Ramsey's theorem for pairs has very recently culminated in Monin and Patey's (in preparation) separation between the stable and general versions of the theorem within standard models of arithmetic. We expect that their new techniques will be widely applicable.

Over the previous decade a more uniform approach to the classification of theorems was developed by Brattka, Gherardi, Marcone and Pauly and many others. Uniformity is achieved in the sense that not just properties of instances are related to properties of solutions, but the dependency of solutions on instances itself is classified. This approach is based on the concept of Weihrauch reducibility and directly considers theorems as mathematical problems that can be compared using the notion of reducibility. This yields a much more fine grained classification of the computational content with purely computability theoretic techniques, the results of which typically refine the results of reverse mathematics.

A very new recent trend is the study of higher areas of Weihrauch complexity that are analogous to ATR_0, a project that is driven by Kihara, Marcone, and Pauly, on the one hand, and Goh on the other.

Workshop (hybrid meeting): Computability Theory

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Proof Mining in Nonconvex Optimization

Ulrich Kohlenbach

During the last two decades a systematic program of 'proof mining' emerged as a new applied form of proof theory and has successfully been applied to a number of areas of core mathematics (see e.g. [5]). Recently, this methodology has been used to extract explicit effective rates of convergence (in the metrically regular case) or rates of metastability in the sense of T. Tao (in the boundedly compact case) for the famous proximal point algorithm (PPA) and its strongly convergent (even in the absence of compactness assumptions) Halpern-type variant (HPPA): see [9, 12, 10, 13, 6, 7, 3, 11, 14] among other papers.

The significance of these algorithms is that they approximate zeros of (maximally) montone operators A which in the case where A (in Hilbert space) is the subdifferential of a lower semi-continuous convex function coincide with the minimizers of the function.

In recent years, nonconvex/nonconcave minimization problems have been studied in optimization. Then the resulting operators A in general will no longer be monotone but may satisfy weaker conditions such as ρ-comonotonicity for negative ρ in the sense of [1] (also called cohypomonotonicity for $|\rho|$ in [2]). In [1], it has been established that (for suitable $\rho \in \mathbb{R}, \lambda > 0$), the resolvent $J_{\lambda A}$ of such operators will be an averaged mapping (but no longer be 1/2-averaged, i.e. firmly nonexpansive). Some control on the averaging constant is sufficient (see [15]) to obtain a common so-called modulus of strong nonexpansivity which is the key concept used in [6, 7]. This makes it possible to generalize the quantitative results obtained in [6, 7] from monotone to ρ-comonotone operators.

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Bi-embeddable categoricity of computable structures

Nikolay Bazhenov

Two structures $\mathcal{A}$ and $\mathcal{B}$ are *bi-embeddable* if $\mathcal{A}$ is isomorphically embeddable into $\mathcal{B}$, and $\mathcal{B}$ is isomorphically embeddable into $\mathcal{A}$. In this talk, we discuss computability-theoretic properties of bi-embeddability types of algebraic structures. For known results on degree spectra up to bi-embeddability, we refer to [1, 2, 3].

Let $\mathbf{d}$ be a Turing degree. A computable structure $\mathcal{S}$ is $\mathbf{d}$-*computably bi-embeddably categorical* (or $\mathbf{d}$-computably b.e. categorical, for short) if for any computable structure $\mathcal{A}$ bi-embeddable with $\mathcal{S}$, there exist $\mathbf{d}$-computable isomorphic embeddings $f\colon \mathcal{A} \hookrightarrow \mathcal{S}$ and $g\colon \mathcal{S} \hookrightarrow \mathcal{A}$. A degree $\mathbf{x}$ is the *degree of bi-embeddable categoricity* for $\mathcal{S}$ if $\mathbf{x}$ is the least degree such that $\mathcal{S}$ is $\mathbf{x}$-computably b.e. categorical.

The paper [4] showed that the index set of $\mathbf{0}'$-computably b.e. categorical, strongly locally finite graphs is Π^1_1-complete. We prove that the index set of computably b.e. categorical graphs is Π^1_1-complete. We also show that every degree $\mathbf{d} \geq \mathbf{0}'$, which contains a Π^0_1 function singleton, is a degree of bi-embeddable categoricity.

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The Cantor-Bendixson theorem in the Weihrauch lattice

Alberto Marcone
(joint work with Vittorio Cipriani, Manlio Valenti)

In a 2015 Dagstuhl seminar I asked "What do the Weihrauch hierarchies look like once we go to very high levels of reverse mathematics strength?" In other words, I proposed to study the multi-valued functions (aka 'problems') arising from theorems which lie around ATR_0 and $\mathbf{\Pi}^1_1\text{-}\mathsf{CA}_0$.

In [3] we started the study of problems arising from statements classically equivalent to ATR_0. In particular, we considered several problems corresponding to the perfect tree theorem. Among these PTT_1 is the problem of finding a perfect subtree of a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ with uncountably many paths. PTT_1 turned out to be equivalent to $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$, the choice function on nonempty closed subsets of Baire space. This, together with other evidence collected in that paper, supports the claim that $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ is one of the problems corresponding to ATR_0: others include $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (the restriction of $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ to singletons) and the problem ATR_2 introduced in [1].

The natural problem corresponding to $\mathbf{\Pi}^1_1\text{-}\mathsf{CA}_0$ is $\mathbf{\Pi}^1_1\text{-}\mathsf{CA} : \mathrm{Tr}^\mathbb{N} \to 2^\mathbb{N}$ (where Tr is the space of subtrees of $\mathbb{N}^{<\mathbb{N}}$): given a sequence of trees $(T_n)_{n\in\mathbb{N}}$ find the characteristic function of $\{n \mid T_n \text{ is well-founded}\}$.

Recall that the Cantor-Bendixson theorem states that any closed set A in a Polish space can be decomposed in the union of its largest (possibly empty) perfect subset (called the perfect kernel of A) and a countable set (called the scattered part of A). The Cantor-Bendixson theorem is well-known to be equivalent to $\mathbf{\Pi}^1_1\text{-}\mathsf{CA}_0$.

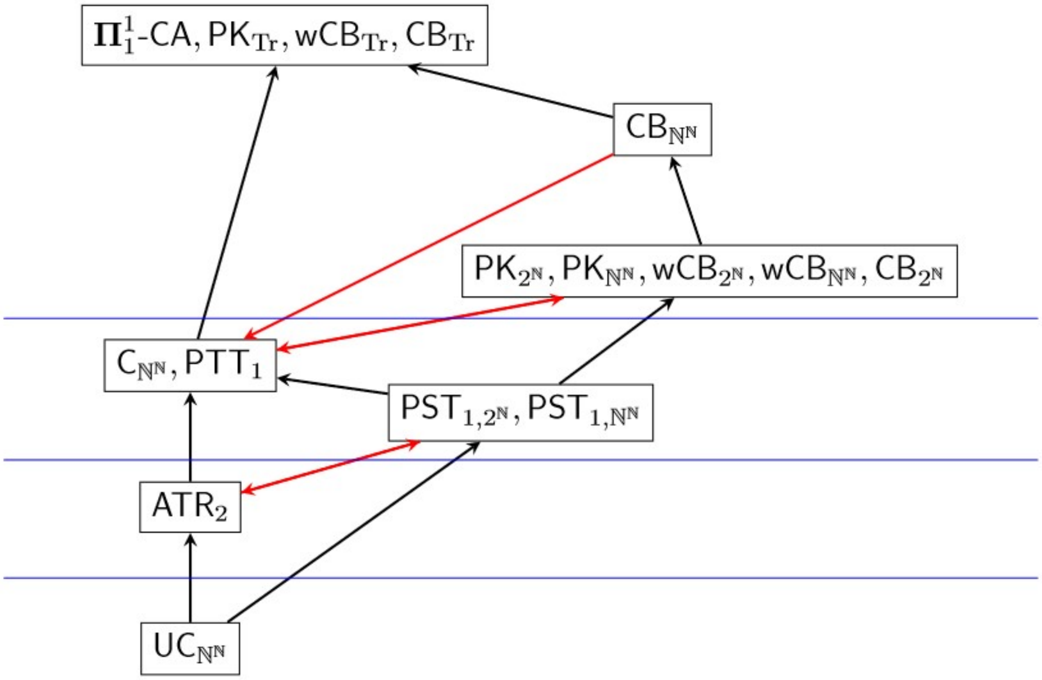
Viewing a closed subset of $\mathbb{N}^\mathbb{N}$ as the set of paths $[T]$ in a tree $T \in \mathrm{Tr}$ we can consider the following problems:

- $\mathsf{PK}_{\mathrm{Tr}} : \mathrm{Tr} \to \mathrm{Tr}$: given a tree T find its largest perfect subtree (called the perfect kernel of T);
- $\mathsf{wCB}_{\mathrm{Tr}} : \mathrm{Tr} \rightrightarrows \mathrm{Tr} \times \mathbb{N}^\mathbb{N}$: given a tree T find its perfect kernel and list the elements of the scattered part of $[T]$;
- $\mathsf{CB}_{\mathrm{Tr}} : \mathrm{Tr} \rightrightarrows \mathrm{Tr} \times \mathbb{N}^\mathbb{N} \times \mathbb{N}$: given a tree T find $\mathsf{wCB}_{\mathrm{Tr}}(T)$ plus the cardinality of the scattered part of $[T]$.

Jeff Hirst in [2] proved that $\mathsf{PK}_{\mathrm{Tr}} \equiv_{\mathrm{W}} \mathbf{\Pi}^1_1\text{-}\mathsf{CA}$, providing the first analysis in the Weihrauch hierarchy of a statement equivalent to $\mathbf{\Pi}^1_1\text{-}\mathsf{CA}_0$.

For X a computable Polish space we consider the following problems:

- $\mathsf{PST}_{1,X} : \subseteq\mathcal{A}^-(X) \rightrightarrows \mathcal{A}^-(X)$: given an uncountable closed subset $A \subseteq X$ find a perfect subset of A;
- $\mathsf{PK}_X : \mathcal{A}^-(X) \to \mathcal{A}^-(X)$: given a closed $A \subseteq X$ find the perfect kernel of A;
- $\mathsf{wCB}_X : \mathcal{A}^-(X) \rightrightarrows \mathcal{A}^-(X) \times \mathbb{N}^\mathbb{N}$: given a closed $A \subseteq X$ find the perfect kernel of A and list the elements of the scattered part of A;
- $\mathsf{CB}_X : \mathcal{A}^-(X) \rightrightarrows \mathcal{A}^-(X) \times \mathbb{N}^\mathbb{N} \times \mathbb{N}$: given a closed $A \subseteq X$ find $\mathsf{wCB}_X(A)$ plus the cardinality of the scattered part of A.



So far we considered only these problems when X is either $\mathbb{N}^\mathbb{N}$ or $2^\mathbb{N}$, but we are planning to expand our research to other spaces, starting with Euclidean spaces. Our results are summarized in the attached figure. Here problems in the same box are Weihrauch equivalent, black arrows mean strict Weihrauch reductions, red arrows mean non-reductions, and blue lines separate arithmetic Weihrauch degrees. The main open problem left is establishing whether $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_{\mathrm{W}} \mathsf{CB}_{\mathbb{N}^\mathbb{N}}$.

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Two Vignettes

Peter Cholak

(joint work with Peter Gerdes, Rod Downey, Noam Greenberg)

We will use our time to hopefully evocatively discuss the two results below. But perhaps a theme which runs though these stories is having fun with effective constructions.

Result 1: Consider the following statement $S(m, n)$: If C is any set which is $(m+1)$-REA and not of m-REA degree, there exists a set A which is n-r.e. in C such that $A \oplus C$ is not of $(m+n)$-REA degree. Soare and Stob [1982] showed this statement holds for $m = 0$ and $n = 1$. Cholak and Hinman [1994] showed that this statement holds for $m = 0, 1$ and arbitrary $n \geq 0$. They also conjectured it holds for all n and m. However, Cholak and Gerdes showed that this statement fails for $n = 2$ and $m = 1$.

Result 2: With Downey and Greenberg, we showed that if a c.e. set A is low_2 then $\mathcal{L}(A)$ and $\mathcal{E}$ are isomorphic.

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Kolmogorov Complexity and Capacitability of Dimension

Theodore Slaman

The Hausdorff Dimension of a set of real numbers A is a numerical indication of the geometric fullness of A. Sets of positive measure have dimension 1, but there are null sets of every possible dimension between 0 and 1.

Effective Hausdorff Dimension is a variant which incorporates computability-theoretic considerations. By work of Jack and Neil Lutz, Elvira Mayordomo, and others, there is a direct connection between the effective Hausdorff dimensions of the elements of a set A and the Hausdorff dimension of A itself. We will describe how this point-to-set principle works and how it allows for novel approaches to classical problems in Geometric Measure Theory.

Namely, in the 1950s Besicovitch and Davies showed that if A is an analytic subset of the real numbers and A has Hausdorff dimension d, then for every s less than d, A has a compact subset of Hausdorff dimension at least s. We show that the assumption that A be analytic cannot be improved within ZFC. Consider the situation under the assumption that $V = L$. Let B be the set of reals x such that x can compute a representation of the ordinal at which x is constructed. As is well-known, B is co-analytic, uncountable and has no uncountable closed subset. A direct point-to-set argument shows that if $V = L$ then B has Hausdorff dimension 1. Consequently, if $V = L$ then the conclusion of the Besicovitch-Davies theorem does not extend to the class of co-analytic sets.

HYP with finite mind-changes: On Kechris-Martin's theorem and a solution to Fournier's question

Takayuki Kihara

According to Steel [2], Kechris and Martin have claimed that the Wadge rank of the ω-th level of the decreasing difference hierarchy of coanalytic sets is ω_2 under the axiom of determinacy. In this talk, we give an alternative proof of the Kechris-Martin theorem, by understanding the ω-th level of the decreasing difference hierarchy of coanalytic sets as the (relative) hyperarithmetical processes with finite mind-changes. Moreover, we give a negative answer to Fournier's question [1] on the gap between the increasing and decreasing difference hierarchies of coanalytic sets, by relating them to the Π^1_1- and Σ^1_1-least number principles, respectively. We also show that the decreasing difference hierarchy of coanalytic sets is a proper subclass of the class of sets which are Δ^1_1 relative to coanalytic sets.

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On definability of c.e. degrees in the 2-c.e. degree structures

Mars Yamaleev

In our talk we will discuss some recent achievements for the problem of definability of c.e. degrees in various degree structures in the finite levels of the Ershov hierarchy. We show how the problem can be solved for the case of m-degrees. Also we analyze possible approaches for the case of Turing degrees, and show how the problem can be solved for easier settings. One of our approaches is based on studying the relative enumerability of 2-c.e. degrees in c.e. degrees below them. Despite an application for the definability questions, the topic of relative enumerability is itself of interest, in particular, there is an old question which goes to the work of Soare and Stob (1982): given a noncomputable low c.e. Turing degree, does there exist a properly 2-c.e. Turing degree which is above it and relatively enumerable in it? We will finish our talk by answering this question.

Relativized depth

Valentino Delle Rose

(joint work with Laurent Bienvenu, Wolfgang Merkle)

The notion of depth was introduced by Bennett in [1], with the aim of separating "useful" and "organized" information from random noise and trivial information. Formally, a subset X of the natural numbers $\mathbb{N}$, identified with its characteristic sequence $X \in \mathbf{2}^{\mathbb{N}}$, is said to be *deep* if for every computable time-bound t,

$$\lim_{n\to\infty} K^t(X \upharpoonright n) - K(X \upharpoonright n) = \infty,$$

where $X \upharpoonright n$ denotes the string formed by the first n bits of the characteristic sequence of X, K is the prefix-free Kolmogorov complexity and K^t denotes its t-time-bounded version. If a set is not deep, we call it *shallow*. A natural example of deep set is given by the halting problem $\emptyset'$. On the other hand, neither ML-random nor computable sets are deep. Finally, depth is upward-closed under tt-reductions:

if X is deep and $X \leq_{tt} Y$, then Y is deep. Since tt-reductions are equivalent to oracle computations running within some computable time-bound and hence, in some sense, "fast", the upward-closure of deep sets under tt-reductions is known as the *slow growth law*, as it expresses, intuitively, that no shallow set is able to compute a deep set fast.

One possible way to relativize the notion of depth, in order to better understand how oracles may help in organizing information, is the following.

Definition. *Given an oracle $A \in \mathbf{2}^{\mathbb{N}}$, a set $X \in \mathbf{2}^{\mathbb{N}}$ is A-*deep *if for every computable time-bound t,*

$$\lim_{n\to\infty} K^{A,t}(X \upharpoonright n) - K^{A}(X \upharpoonright n) = \infty,$$

where K^A and $K^{A,t}$ denotes, respectively, the unbounded and t-time-bounded prefix-free complexities relative to A.

The main properties of depth relativize in the following way: neither A-ML-random nor A-tt-computable sets are deep, A' is A-deep and, finally, if A is A-deep and $X \leq_{tt} Y \oplus A$, then Y is A-deep.

Intuitively speaking, access to an oracle increases computation power and accordingly, for most classes of sets that are considered in computability theory the relativized version either contains or is contained in the unrelativized version, examples are given by the classes of computable and of ML-random sets, respectively. However, this is not the case of depth. Indeed, given an oracle A, the four following cases may occur.

(1) **Depth and A-depth may be incomparable**. We show that this is the case for $\emptyset'$. Indeed, $\emptyset'$ is clearly $\emptyset'$-shallow. Moreover, the following holds.

Theorem 1. *There exists a Δ^0_2 set X which is ML-random (hence shallow) but $\emptyset'$-deep.*

(2) **A-depth may be strictly weaker than depth**. That is, every deep set is A-deep and there are shallow sets which are A-deep. We show that this is the case whenever A is ML-random. In particular, we prove the following results.

Theorem 2. *If X is deep and A is ML-random, then X is A-deep.*

A weaker statement is also true for shallowness.

Theorem 3. *If X is shallow and A is X-2-random, then X is A-shallow.*

However, Theorem 3 fails in the unrelativized case. Since complete extensions of PA are deep (see [2]), the Randomness Basis Theorem ([3], Prop. 7.4) guarantees the existence of a deep set Y such that A is Y-ML-random. Then the set $X = (Y \operatorname{xor} A)$, i.e. the symmetric difference of Y and A, is both Y-ML-random (hence shallow) and A-deep.

Theorem 4. *For every ML-random oracle A there exists a shallow set X which is A-deep.*

Such a set X can be also used to give a short proof of the fact that every PA-complete degree is the join of two ML-random degrees, which has been shown in [4].

(3) **A-depth may be strictly stronger than depth.**

(4) **A-depth may coincide with depth.**

It is easy to see that every K-trivial oracle falls either under case (3) or case (4). However, it is open in general which of these two cases may apply. Obviously, case (4) apply to all computable sets.

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Generic algebraic fields

Russell Miller

(joint work with Kirsten Eisenträger, Caleb Springer & Linda Westrick)

The goal of this work is to investigate the "common" behavior of algebraic field extensions of the rational numbers $\mathbb{Q}$, with respect to Hilbert's Tenth Problem and related questions. Such fields are exactly the subfields of (a fixed computable presentation of) the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. We use $\mathrm{Sub}(\overline{\mathbb{Q}})$ to denote the space of all those subfields. We are justified in calling it a space, because it has an established topology, under which it is homeomorphic to the Cantor space 2^ω: the clopen sets, which form a basis, are precisely those sets of the form

$$\mathcal{U}_{\overline{a};\overline{b}} = \{F \in \mathrm{Sub}(\overline{\mathbb{Q}}) \ : \ \mathbb{Q}(\overline{a}) \subseteq F \ \& \ \overline{b} \cap F = \emptyset\},$$

for finite tuples $\overline{a}$ and $\overline{b}$ from $\overline{\mathbb{Q}}$. This topology is recognizable to field theorists: it arises under a standard construction that dates back to Vietoris. A separate development given in [5] considers the space of all isomorphism types of algebraic field extensions of $\mathbb{Q}$, which is just the quotient space modulo isomorphism of the space $\mathrm{Sub}(\overline{\mathbb{Q}})$, and which too is homeomorphic to 2^ω. Since 2^ω has the property of Baire, so do both of these spaces, and we therefore naturally use the notion of a meager subset of $\mathrm{Sub}(\overline{\mathbb{Q}})$ to describe size: a property P of fields is "common" if the collection of subfields of $\overline{\mathbb{Q}}$ satisfying P is comeager. Here we will state results for $\mathrm{Sub}(\overline{\mathbb{Q}})$, but analogous results also hold in the space of isomorphism types.

For a field F (or any ring), *Hilbert's Tenth Problem for F* is the set

$$\mathrm{HTP}(F) = \{f \in F[X_1, X_2, \ldots] \ : \ f = 0 \text{ has a solution in } F\},$$

describing the polynomial equations over F that can be solved in F. The *root set of F* is its restriction to single-variable polynomials

$$R_F = \{f \in R[X_1] \ : \ f \in \mathrm{HTP}(F)\}.$$

Hilbert's original Tenth Problem, from 1900, was the case $F = \mathbb{Z}$, and was resolved in 1970 by Matiyasevich [4], who used work [1] of Davis Putnam, and Robinson to show that $\mathrm{HTP}(\mathbb{Z})$ is computably isomorphic to the Halting Problem. The decidability status of $\mathrm{HTP}(\mathbb{Q})$ remains unknown. At the other extreme in $\mathrm{Sub}(\overline{\mathbb{Q}})$, however, $\mathrm{HTP}(\overline{\mathbb{Q}})$ is trivially decidable. Our purpose is to investigate the Turing degrees of $\mathrm{HTP}(F)$ for fields F algebraic over $\mathbb{Q}$, using techniques from [2].

For the restricted case of R_F, this investigation begins with Rabin's Theorem. Rabin's original theorem appeared in [8], as a result about computable fields. Here we give a relativized version, applicable to all presentations F (computable or not) of algebraic subfields of $\overline{\mathbb{Q}}$. We define a *Rabin embedding* of F (into our fixed presentation $\overline{\mathbb{Q}}$) to be a field homomorphism $f : F \to \overline{\mathbb{Q}}$ that is computable in the atomic diagram $\Delta(F)$. Its image $f(F)$ is a *Rabin image* of F.

Theorem 1 (Rabin). *There is a Turing functional Φ such that, for every presentation F of any algebraic field extension of $\mathbb{Q}$, $\Phi^{\Delta(F)}$ is a Rabin embedding of F into $\overline{\mathbb{Q}}$. Moreover, for every Rabin embedding f of F, we have Turing equivalence between R_F and the image $f(F)$, relative to $\Delta(F)$:*

$$f(F) \oplus \Delta(F) \equiv_T R_F \oplus \Delta(F),$$

with both Turing reductions given uniformly in f.

Thus a Rabin image $f(F)$ of F need not be $\Delta(F)$-computable, since R_F may not be computable from $\Delta(F)$. Of course, $f(F)$ is always c.e. relative to $\Delta(F)$, as is R_F. It is also useful here to introduce the *index set* I_F of a presentation F:

$$I_F = \{f \in \mathbb{Z}[X] \ : \ f \text{ has a root in } F\}.$$

The polynomials here have coefficients in a fixed copy $\mathbb{Z}$ of the integers, independent of the presentation. The index set identifies the isomorphism type: $I_E = I_F$ just if $E \cong F$. Again, I_F is always c.e. in $\Delta(F)$, but need not be computable from $\Delta(F)$. In fact, $I_F \oplus \Delta(F) \equiv_T R_F \oplus \Delta(F)$, with uniformity in both reductions.

The essence of the following theorem appeared first in [6].

Theorem 2. *Let $F \in \mathrm{Sub}(\overline{\mathbb{Q}})$ be a generic subfield. Then there exists a presentation $E \cong F$ such that $I_F = I_E \not\leq_T \Delta(E)$ (and hence $R_E \not\leq_T \Delta(E)$ too). However, every $E \cong F$ satisfies*

$$(I_E)' \leq_T (R_E \oplus \Delta(E))' \leq_T (\Delta(E))'.$$

In short, R_E and I_E are always low relative to $\Delta(E)$, but each sometimes fails to be computable in $\Delta(E)$.

This contrasts with the situation for fields in $\mathrm{Sub}(\overline{\mathbb{Q}})$ generally. Many fields – for example, all number fields F – have $R_F \leq_T \Delta(F)$ (and since number fields all have computable presentations, I_F is decidable). On the other hand, other fields

K have $R_K \equiv_1 (\Delta(K))'$, so that R_K and I_K are non-low, indeed as complex as possible, relative to $\Delta(K)$. In fact, for each c.e. set W of prime numbers, the field $\mathbb{Q}(\sqrt{p}\ :\ p \in W)$ has a computable presentation K with $I_K \equiv_T R_K \equiv_T W$. So, while Theorem 2 describes the generic situation, each aspect described (noncomputability and lowness) can definitely fail for non-generic fields.

The main theorem presented in the talk at Oberwolfach on April 27 states that, for generic fields F (and hence on a comeager subset of $\text{Sub}(\overline{\mathbb{Q}})$), the multivariable case is no more difficult than its restriction to the single-variable case. The key to its proof, arising from [2], is a demonstration that a natural forcing relation $(\overline{a}; \overline{b}) \Vdash \alpha$ is decidable for all existential sentences α in the language of fields.

Theorem 3 (Eisenträger-Miller-Springer-Westrick)**.** *Let $F \in \text{Sub}(\overline{\mathbb{Q}})$ be a generic subfield. Then every presentation $E \cong F$ has $R_E \oplus \Delta(E) \equiv_T HTP(E) \oplus \Delta(E)$. Thus, among presentations E of F, Hilbert's Tenth Problem $HTP(E)$ must sometimes be noncomputable relative to $\Delta(E)$, but will always be low relative to $\Delta(E)$.*

In view of Rabin's Theorem, we can add a corollary.

Corollary 4. *There is a uniform procedure that, for every generic subfield $E \subseteq \overline{\mathbb{Q}}$, decides $HTP(E)$ when given an oracle for E as a subset of $\overline{\mathbb{Q}}$.*

We emphasize the distinction: in Theorem 3, we are given an algebraic field as a freestanding structure, whereas in Corollary 4, the oracle tells us exactly which elements of $\overline{\mathbb{Q}}$ lie in E and which do not. By Rabin's Theorem, the former situation is equivalent to having an enumeration of the subfield of $\overline{\mathbb{Q}}$, while in the latter, the oracle serves as a decision procedure for membership in the subfield. To understand why the latter is more powerful, imagine the simple question of whether $X^3 - 2$ has a root in the subfield. Given a decision procedure, we simply find the three cube roots of 2 in $\overline{\mathbb{Q}}$ and use the procedure to check whether any of them lies in the subfield. However, from an enumeration of the subfield (or in a freestanding presentation of the field), we can recognize a cube root of 2 if one ever appears, but can never be sure that no such cube root lies in the subfield: the enumeration might simply not have revealed it yet.

It would be natural to try to transfer Lebesgue measure from Cantor space to $\text{Sub}(\overline{\mathbb{Q}})$, and this would yield a different notion of size of subsets of $\text{Sub}(\overline{\mathbb{Q}})$, possibly giving different answers about what behaviors are "common." The difficulty with this approach is that the transferred measure depends heavily on the choice of the homeomorphism used for the transfer, and there is as yet no natural choice known. (In particular, the *Haar-compatible measure* defined in [5] is not as simple as was reported there, and has been abandoned for the present.) We still hope to determine an appropriate probability measure on $\text{Sub}(\overline{\mathbb{Q}})$ for these purposes, but for now this question remains open. Baire category, of course, avoids these problems, as the concept of meagerness is defined in purely topological terms: thus a subset of $\text{Sub}(\overline{\mathbb{Q}})$ has comeager preimage in 2^ω under one homeomorphism just if its preimage there is comeager under every homeomorphism.

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Combinatorial equivalence of a computability theory question

Lu Liu

We prove that a question of Miller and Solomon—whether every coloring c on the d-ary string space admits a c-computable variable word infinite solution, is equivalent to a combinatorial question. The combinatorial question asks whether there is an infinite sequence of integers so that each of its prefixes satisfies a Ramsey type property. The negation of the combinatorial question is a generalization of the Hales-Jewett theorem (a cornerstone in combinatorics).

On the "Heap" Problem

Vadim Puzarenko
(joint work with Smeliansky R.L.)

There is the deep relation between well-known paradox "What the heap is?" (belonging to Eubulid from Miletus) and certain problems of quantum mechanics. As Maslov V.P. is said in [1], "If we measure the quantity of grains of sand by counting them – it's not a heap. If we don't want or can not to count them and measure by other means – it's a heap. . . If from an open bottle with gas which was considered by L. Boltzmann, 10^6 particles goes out in 10^{-10} seconds, it is impossible to number these particles in order to trace their further path". However, the quantitative justification of impossibility isn't given anywhere in that paper.

To sort 10^6 molecules in 10^{-10} seconds, we need to execute about $6 \cdot 10^6$ comparison operations for this time. It means that, even taking into account a possibility of parallelizing merge operations of merge in Hoare's sorting, our computer must have the clock frequency about $17 \cdot 10^{18}$ Hz. But this frequency corresponds to a

X-ray radiation! Without going into technical details of the modern and perspective technologies in the field of an electronics engineering, the computer with such clock frequency can't exist.

We give a new algorithm for L. Boltzmann's task.

Theorem 1 ([2]). *There exists an algorithm for Boltzmann's task whose level complexity is $O(n)$. Moreover, it depends on shot but is independent of the quantity of particles.*

Notice that the considered shot is fixed. How many particles we can number within 10^{-10} s, if the maximum frequency of a modern processing unit is no more than $5 \cdot 10^9$ Hz and the comparison operation takes 1 clock tick? It is not difficult to calculate that the answer is none! With the modern level of computing technology, the goal is not reachable.

Finally, we introduce additional computational structures for L. Boltzmann's task.

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Ill-founded orders and Weihrauch degrees

Manlio Valenti

(joint work with Jun Le Goh, Arno Pauly)

In this work we investigate the uniform strength of the two (Weihrauch equivalent) problems DS and BS, where the first one consists in finding a descending sequence through an ill-founded linear order, and the second one asks for a bad sequence through a non-well quasi-order. The problem DS can be seen as a "one-sided" version of ADS (given an infinite linear order produce either an ascending or a descending chain). However their computational contents are very different: while the latter is easily seen to be Weihrauch reducible to RT^2_2, and hence is an arithmetic problem, there is a computable ill-founded linear order with no hyperarithmetic descending sequence, which places DS in the "non-hyperarithmetic" part of the Weihrauch hierarchy.

Recently [2] introduced the concept of first-order part ${}^1f$ of a multi-valued function f, intuitively describing the strongest problem with codomain $\mathbb{N}$ reducible to f. In the same spirit, we define the notion of deterministic part $\mathrm{Det}_Y(\cdot)$ of a multi-valued function, representing the strongest (single-valued) function with codomain Y reducing to f. We show that ${}^1\mathsf{DS} \equiv_{\mathrm{W}} \mathbf{\Pi}^1_1{-}\mathsf{Bound}$, where $\mathbf{\Pi}^1_1{-}\mathsf{Bound}$ is the problem that takes in input a finite $A \in \mathbf{\Pi}^1_1(\mathbb{N})$ and returns a bound for it. We also show that $\mathrm{Det}_{\mathbb{N}^\mathbb{N}}(\mathsf{DS}) = \mathsf{lim}$. We further characterize the lower cone of DS under Weihrauch reducibility, showing how it misses many arithmetic problems.

We also consider the problems $\mathbf{\Gamma}$-DS and $\mathbf{\Gamma}$-BS of finding descending sequences (resp. bad sequences) through a $\mathbf{\Gamma}$ presented linear order (resp. quasi-order), where $\mathbf{\Gamma} \in \{\mathbf{\Sigma}^0_k, \mathbf{\Pi}^0_k, \mathbf{\Delta}^0_k, \mathbf{\Sigma}^1_1, \mathbf{\Pi}^1_1, \mathbf{\Delta}^1_1\}$.

This creates a hierarchy of DS-like problems. We show that this hierarchy does not collapse at any finite level by characterizing the first-order part of $\mathbf{\Delta}^0_k$-DS. In particular, for each k we have $\mathbf{\Sigma}^0_{k+1}$-$\mathsf{DS} \equiv_{\mathrm{W}} \mathbf{\Delta}^0_{k+1}$-$\mathsf{BS} \equiv_{\mathrm{W}} \mathbf{\Delta}^0_{k+1}$-$\mathsf{DS} \equiv_{\mathrm{W}} \mathbf{\Pi}^0_k$-$\mathsf{DS}$.

We also exploit and generalize a technique based on Π^1_1-inseparable sets (first used in [1]) to show that $\mathbf{\Sigma}^1_1$-DS is strictly weaker than the problem $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (which can be thought of as the problem of finding a path through an ill-founded subtree of $\mathbb{N}^{<\mathbb{N}}$). The problems $\mathbf{\Pi}^1_1$-DS and $\mathbf{\Sigma}^1_1$-BS are much stronger: they can be used to compute the leftmost path of an ill-founded tree, and this locates them in the realm of $\mathbf{\Pi}^1_1$−CA_0.

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Densely computable structures

Valentina Harizanov

(joint work with Wesley Calvert, Douglas Cenzer)

In 2012, Jockusch and Schupp studied generically computable and coarsely computable sets, which they defined using dense sets (see [6, 5]). We generalize these notions of approximate computability from sets to structures. The *asymptotic density* of a set $A \subseteq \omega$, if it exists, is

$$\lim_{n\to\infty} \frac{|A\cap\{0,\dots,n\}|}{n+1}.$$

We say that a set A is *dense* if its asymptotic density is 1.

We show that a set $A \subseteq \omega$ has asymptotic density δ if and only if the set $A \times A$ has density δ^2 in $\omega \times \omega$. We also show that there is a computable dense set $C \subseteq \omega \times \omega$ such that for any infinite c.e. set A, the product $A \times A$ is not a subset of C. These results lead us to the notion of a generically computable structure below [1]. We further define a Σ_n generically c.e. structure using the following definition of a Σ_n elementary substructure. We say that a substructure $\mathcal{B}$ of $\mathcal{A}$ is a Σ_n *elementary substructure* if for every infinitary Σ_n formula $\theta(x_1,\dots,x_n)$ and any $b_1,\dots,b_n$ from $\mathcal{B}$, we have:

$$\mathcal{A} \models \theta(b_1,\dots,b_n) \text{ iff } \mathcal{B} \models \theta(b_1,\dots,b_n).$$

Consider structures for finite languages. A structure is computable if its domain is a computable set and its relations and functions are computable. We say that a structure $\mathcal{D}$ with domain D is a c.e. structure if D is c.e., each relation of $\mathcal{D}$ is c.e., and each function of $\mathcal{D}$ is the restriction of a partial computable function to D. By c_R we denote the characteristic function of R.

Definition 1.

(1) A structure $\mathcal{A}$ is *generically computable* if $\mathcal{A}$ has a substructure $\mathcal{D}$ with a c.e. domain D of asymptotic density 1 such that for every k-ary function f and every k-ary relation R of $\mathcal{A}$, both $f \restriction D^k$ and $c_R \restriction D^k$ are restrictions to D^k of some partial computable functions.

(2) A structure $\mathcal{A}$ is Σ_n *generically c.e.* if there is a c.e. dense set D such that the substructure $\mathcal{D}$ with domain D is a c.e. substructure and also a Σ_n elementary substructure of $\mathcal{A}$.

We will now focus on injection structures (see also [4]) and equivalence structures (see also [2, 3]). An *injection structure* $\mathcal{A}$ is a set A together with a one-to-one function $f : A \to A$. It is not hard to see that every c.e. injection structure is isomorphic to a computable injection structure.

The *orbit* of an element a under f is

$$\mathcal{O}_f(a) = \{x : (\exists n \in \omega)[x = f^{(n)}(a) \ \vee \ a = f^{(n)}(x)]\}.$$

Infinite orbits may be of type $\mathbb{Z}$ or of type ω. The *character* of $\mathcal{A}$ is

$$\chi(\mathcal{A}) = \{(k, n) \in (\omega \setminus \{0\}) \times (\omega \setminus \{0\}) : \mathcal{A} \text{ has at least } n \text{ orbits of size } k\}.$$

Theorem 2. *An injection structure $\mathcal{A} = (\omega, f)$ has a generically computable copy iff*

(i) *$\mathcal{A}$ has an infinite substructure isomorphic to a computable structure iff*

(ii) *$\mathcal{A}$ has an infinite orbit or $\chi(\mathcal{A})$ has an infinite c.e. subset.*

Having a Σ_1 generically c.e. isomorphic copy has a simple characterization.

Theorem 3. *A structure $\mathcal{A} = (\omega, f)$ has a Σ_1 generically c.e. copy iff*

(i) *$\mathcal{A}$ has a computable copy iff*

(ii) *$\chi(\mathcal{A})$ is a c.e. set iff*

(iii) *$\mathcal{A}$ has a Σ_2 generically c.e. copy.*

An *equivalence structure* $\mathcal{A} = (A, E)$ is a set A with an equivalence relation E on A. Its character is denoted by $\chi(\mathcal{A})$. We have a surprising result that every equivalence structure (ω, E) has a generically computable copy. We show that if an equivalence structure (ω, E) is generically computable, then there is an infinite computable $C \subseteq \omega$ such that the restriction of E to $C \times C$ is computable.

Theorem 4. *An equivalence structure $\mathcal{A} = (\omega, E)$ has a Σ_1 generically c.e. copy if and only if at least one of the following conditions holds:*

(a) *$\chi(\mathcal{A})$ is bounded;*

(b) *$\chi(\mathcal{A})$ has a Σ^0_2 subset K which is a character with a computable Khisamiev s_1-function;*

(c) *$\mathcal{A}$ has an infinite class and $\chi(\mathcal{A})$ has a Σ^0_2 subset K;*

(d) *$\mathcal{A}$ has infinitely many infinite classes.*

Theorem 5. *A structure $\mathcal{A} = (\omega, E)$ has a Σ_2 generically c.e. copy iff*

(i) *$\mathcal{A}$ has a c.e. copy iff*
(ii) *$\mathcal{A}$ has a Σ_3 generically c.e. copy.*

Next, we introduce the notions of coarsely computable and coarsely Σ_n structures [1].

Definition 6.

(1) A structure $\mathcal{A}$ is *coarsely computable* if there are a computable structure $\mathcal{E}$ and a dense set D such that the structure $\mathcal{D}$ with domain D is a substructure of both $\mathcal{A}$ and $\mathcal{E}$ and all relations and functions agree on D.
(2) A structure $\mathcal{A}$ is Σ_n *coarsely c.e.* if there are a c.e. structure $\mathcal{E}$ and a dense set D such that the substructure $\mathcal{D}$ with domain D is a Σ_n elementary substructure of both $\mathcal{A}$ and $\mathcal{E}$ and all relations and functions agree on D.

We show that every generically computable injection structure has a coarsely computable copy, while there is a generically computable injection structure that is not coarsely computable. There is a coarsely computable injection structure with no generically computable copy.

Theorem 7. (1) *There are injection structures with no coarsely computable copies.*
(2) *There are equivalence structures with no Σ_1 coarsely c.e. copies.*

Theorem 8. *An injection structure $\mathcal{A} = (\omega, f)$ has a Σ_1 coarsely c.e. copy iff*

(i) *$\mathcal{A}$ has a computable copy iff*
(ii) *$\chi(\mathcal{A})$ is a c.e. set.*

We also characterize equivalence structures with Σ_2 coarsely c.e. copies, as well as Σ_3 coarsely c.e. equivalence structures.

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Almost Theorems of Hyperarithmetic Analysis

Richard A. Shore

Theorems of hyperarithmetic analysis (THA) occupy an unusual neighborhood in the realms of reverse mathematics and recursion theoretic complexity. They lie above all the fixed (recursive) iterations of the Turing Jump but below ATR_0 (and so Π^1_1-CA_0 or the hyperjump). There is a long history of proof theoretic principles which are THA. Until Barnes, Goh and Shore [ta], there was only one mathematical denizen found by Montalbán [2006]. BGS studied many variations of graph theoretic theorems of Halin [1965], [1970] and others appearing in Diestel's standard text [2017]. This study revealed an array of theorems in graph theory living in this THA neighborhood.

We introduce a new neighborhood of theorems which are almost theorems of hyperarithmetic analysis (ATHA). When combined with ACA_0 they are THA but on their own the are very weak. We generalize several conservativity classes (Π^1_1, r-Π^1_1 and Tanaka) and show that all our examples (and many others) are conservative over RCA_0 in all these senses and weak in other recursion theoretic ways as well. We provide denizens both mathematical and logical.

The original motivating result for this investigation was what appeared to be a reduction between two variants of the theorems studied in BGS. This reduction appeared in a graph theoretic paper by Bowler, Carmesin and Pott [2015]. It relied on the fact that every graph satisfying the hypotheses of Halin's theorem has a locally finite subgraph also satisfying these conditions. This theorem turned out to be an ATHA. We then found many other variations of standard theorems and logical theories which are all highly conservative over RCA_0 but very strong over ACA_0. They include examples going up a hierarchy which ends at systems very weak over RCA_0 but stronger than full second order arithmetic over ACA_0.

These results can be seen as answering a question raised by Hirschfeldt and reported in Montalbán [2001] by providing a long list of pairs of principles one of which is very weak over RCA_0 but over ACA_0 is equivalent to the other which may be strong (THA) or even much stronger. Thus one can say that they supply a collection of theorems and theories which should be considered relative to ACA_0 rather than RCA_0.

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On A-computable Numberings

Marat Faizrahmanov

This talk is concerned with the computational properties of families of subsets of $\mathbb{N}$. Recall that any surjective mapping from $\mathbb{N}$ onto a countable family $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a *numbering* of $\mathcal{F}$. Following [1, 2], we say that, for a given set A, a numbering ν is *A-computable* if there is a computable function f such that $\nu(x) = W^A_{f(x)}$ for each $x \in \mathbb{N}$. Families with A-computable numberings are also called *A-computable.* A numbering ν is said to be *reducible* to a numbering μ ($\nu \leqslant \mu$) if $\nu = \mu \circ f$ for some computable function f. Two numberings ν and μ are said to be *equivalent* ($\nu \equiv \mu$) if they are reducible to each other. *The Rogers semilattice* of an A-computable family is the quotient structure of its A-computable numberings with respect to the equivalence of numberings.

The Rogers semilattice of a computable family can be viewed as an algebraic reflection of its effective topological properties. For example, the Rogers semilattice of any computable effectively discrete family is one-element, but the Rogers semilattice of any finite family with two sets comparable under inclusion is infinite.

The first part of the talk presents results on Rogers semilattices of A-computable families for any non-computable oracle A. Namely, we consider the questions about their possible cardinalities, their latticeness, their distributivity, existence of their largest and minimal elements, etc.

In the second part of the talk, we consider semilattices of all computable families and all A-computable families under set-theoretic inclusion, where A is an arbitrary oracle. These semilattices were introduced and first studied by A.N. Degtev [3]. We consider the questions about definability of their Fréchet ideals, about existence of their nontrivial definable singletons, and formulate some related questions.

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Randomness relative to almost everything

Tomasz Steifer
(joint work with Laurent Bienvenu, Valentino Delle Rose)

A famous theorem by van Lambalgen [2] states that for $A \oplus B$ is Martin-Löf random if and only if A is Martin-Löf random and B is Martin-Löf relative to A, if and only if B is Martin-Löf random and A is Martin-Löf random relative to B. It follow by an easy argument that every Martin-Löf random set X is also Martin-Löf random relative to almost all oracles. We introduce a following template notion:

Definition 1. *Given a notion of relativized () randomness, we will say that X is a.e. () random if the set of oracles Z such that X is not () random relative to Z, is of measure zero.*

It is known that for some notions of randomness a version of van Lambalgen's theorem does not hold. For instance, it was already observed by Yu [3] that one of the implications fails for relativized computable randomness. Our main observation is that van Lambalgen's theorem fails for randomness in a certain strong sense, namely that

Theorem 2. *There exist a sequence X which is computably random but not a.e. computably random.*

The proof uses the fireworks technique advanced recently by several authors. At the same time, Kolmogorov complexities of initial segments of an a.e. computably random may be small—too small for a Martin-Löf random or even even a partial computably random. This separation happens in almost everywhere dominating degrees introduced by Dobrinen and Simpson [1].

Some open problems remains. We do not know if there is a set which is partial computably random but not a.e. computably random. We do not have a proof of separation between partial computable randomness and a.e. partial computable randomness. More results will follow.

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Effectively Hausdorff Spaces

Matthias Schröder

In a previous Oberwolfach Workshop on Computability Theory, A. Pauly posed the question whether or not any computably compact computably Hausdorff space is computably regular (cf. [1, 5, 4]). We show that the answer to this question is no. Indeed, the one-point compactification of any computable metric space that is not locally compact yields a counterexample.

As a remedy, we introduce the new notion of an effectively Hausdorff space which is stronger than the aforementioned notion of a computable Hausdorff space. A computably admissible space X is called *effectively Hausdorff*, if there are two computable sequences $(u_i)_{i\in\mathbb{N}}, (v_i)_{i\in\mathbb{N}}$ of open subsets of X such that

$$\bigcup_{i\in\mathbb{N}}(u_i \times v_i) = \{(x,y) \in X \times X : x \neq y\}.$$

This new notion includes computable metric spaces and admits the effectivisation of some classical theorems from topology. In particular, the answer to the above question changes to yes. A computably compact effectively Hausdorff space forms even a computable metric space, provided that it has a computable dense sequence.

Furthermore, we discuss a form of compact overt choice that is computable for any effectively Hausdorff space. Compact overt choice means the computational task of finding an element in a non-empty compact subset given by positive information (see [3] for closed overt choice). This result can be used to prove a characterisation of computable multifunctions.

Theorem 1. *Let X be a computable metric space, and let Y be an effectively Hausdorff space. Then a total multifunction $F\colon X \rightrightarrows Y$ is computable if, and only if, there are computable functions $h_+\colon X \to \mathcal{K}_+(Y)$ and $h_b\colon X \to \mathcal{K}_-(Y)$ satisfying*

$$\emptyset \neq h_+(x) \subseteq F[x] \cap h_b(x)$$

for all $x \in X$.

Here $\mathcal{K}_+(Y)$ and $\mathcal{K}_-(Y)$ denote the represented spaces of compact subsets of Y equipped with the positive/negative representation for the compact subsets, respectively (cf. [5, 6]). This result generalises a similar result by V. Brattka and P. Hertling in [2] for the case that both spaces X, Y are computable metric spaces. An open problem is to find a corresponding characterisation of computable multifunctions for the case that both spaces X, Y are effectively Hausdorff spaces.

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Coarse computability, the density metric, and Hausdorff distances between Turing degrees

Denis R. Hirschfeldt
(joint work with Carl G. Jockusch, Jr., Paul E. Schupp)

We discuss the metric space of coarse similarity classes of sets of natural numbers under the distance function given by the upper density of the symmetric difference of two sets. We define a notion of distance between Turing degrees based on Hausdorff distance in this space, and discuss aspects of the resulting metric space, which by a relativized version of a result of Monin, is $(0, 1/2, 1)$-valued. We also discuss computability-theoretic aspects of a Ramsey-theoretic theorem due to Mycielski with connections to algorithmic randomness.

PA relative to an enumeration oracle

Mariya Soskova
(joint work with Jun Le Goh, Iskander Kalimullin, Joseph S. Miller)

Relativization is an important tool used in computability theory. It allows us to extend properties or relations defined for computable sets to arbitrary Turing oracles, by replacing the computably enumerable (c.e.) component of the definition by c.e. relative to the Turing oracle. We can extend these definitions further to capture relations between enumeration oracles by replacing "c.e. in" by "enumeration reducible to".

We study the relation "PA relative to an enumeration oracle". We isolate several classes of enumeration oracles based on their behavior with respect to this relation: the PA bounded oracles, the oracles that have a universal class, the low for PA oracles, the self-PA oracles. We study the relationship between them and other known classes. We also investigate a group of classes of oracles that were introduced by Kalimullin and Puzarenko [1] based on properties that are commonly studied in descriptive set theory. We characterize most of these classes by restricting the relation "PA above" to a special sub-collection of Π^0_1 classes. This allows us to completely determine the relative position of all classes in question.

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Complexity of lines and planes: finite case

Alexander Shen

Consider a line on the plane $\mathbb{R}^2$ that has effective Hausdorff dimension s. (A dimension of a line $y = ax + b$ is the dimension of the pair (a, b); the dimension of a line $x = c$ is the dimension of c. We omit here the words "effective Hausdorff" since we do not consider other dimensions.) Then every point on the line has dimension at most $1 + s$, since this point is determined by the line and one of the point's coordinates. Also the dimension of every point does not exceed 2. Lutz and Stull have recently proven that these two upper bounds are the only ones: some of the points have dimension $\min(1 + s, 2)$.

To understand this result better, it is natural to look at its finite version. The effective Hausdorff dimension is defined as lim inf of the complexities of prefixes (per bit), so one could ask what happens with these complexities *without* lim inf.

First, one can look at a finite field case. Assume that $\mathbb{F}$ is a finite field of size about 2^n. The typical complexity of its elements is about n. (We consider complexities with $O(\log n)$ precision.) Consider the affine plane $\mathbb{F}^2$; lines on this plane have typical complexity $2n$, and points also have typical complexity $2n$. The following finite version of Lutz–Stull theorem holds:

> *If a line has complexity s, then the maximal complexity of points on it is* $\min(2n, s + n) + O(\log n)$.

The combinatorial translation of this statement: *if a set contains 2^t points for $t < n - O(1)$, then at most 2^{t-n} lines can be its subsets.* This is easy to show, for example, using the inclusion-exclusion formula (or the expander properties of the lines–points graph).

Second, one can consider the continuous case, looking at the 2^{-n}-approximations of points and line coefficients. However, there is a technical problem, since the line is infinite and this implies that the maximal complexity of (an approximation) of its point is unbounded. To avoid this problem, one can consider the projective plane, a two-dimensional compact manifold (a sphere with identified opposite points). The lines are great circles on this sphere, and can be identified with corresponding poles (duality). So we can speak about the complexity of 2^{-n}-approximation of points and lines. (Similar notion of approximation complexity can be defined for most natural compact manifolds.)

It is quite surprising, taking into account Lutz–Stull result, that the bound for the finite approximation is much weaker:

> *If a line on the projective plane has 2^{-n}-approximation complexity s, then it contains a point whose 2^{-n}-approximation complexity is $n + s/2$.*

Again there is a combinatorial counterpart connecting the area of a set A on the projective plane (or on the sphere) with the area of set B of all poles of great circles that lie entirely in A. The following geometric inequality holds:

> *Let A and B be two measurable sets on the sphere of unit area. Assume that for every $b \in B$ all the points a that are orthogonal to b belong to A. Then $\mu(B) \leq \mu(A)^2$.*

Here μ is the uniform measure on the sphere S and $\mu(S) = 1$.

Indeed, let ξ and η be independent random variables uniformly distributed on S, and let ζ be the (unique) vector that is orthogonal to ξ and η, and forms a positively oriented triple with ξ and η. (This vector is undefined when $\xi = \eta$, but this happens with probability 0.) Then, for symmetry reasons, ζ is uniformly distributed on S, so the event $\zeta \in N$ has probability $\mu(B)$. On the other hand, it is a subset of the event $(\xi \in A) \wedge (\eta \in A)$ that has probability $\mu(A)^2$ due to independence.

It is easy to see that the bound $s/2 + n$ is achieved for the case where the line is randomly taken from a small neighborhood of a simple line. This means that the result of Lutz and Stull essentially uses the $\lim inf$: the prefix where the $\liminf$ is achieved for the line can be shorter than the prefix for the point where the $\liminf$ is achieved.

Open question: How to formulate and prove the corresponding geometric statement?

Acknowledgments: this open question arises from the discussions with Penn State Logic Seminar (in particular, with Linda Westrick) and Moscow University Kolmogorov seminar. The proof of the inequality for sphere areas was communicated by Ilya Bogdanov (and simplified by Alexander Kozachinskiy).

Open Problems

Workshop participants

(Nikolay Bazhenov) Is the class of Heyting algebras universal in the sense of the paper of Hirschfeldt, Khoussainov, Shore, and Slinko (2002)? One can also consider restricted versions of this question: for example, can computable Heyting algebras realize all possible categoricity spectra?

(Verónica Becher) If we toss a coin N times, the uniform probability distribution says that each block of length n will occur about $N/2^n$ times. This is informative only in case $n \leq \log N$. What can we say about blocks of length larger than $\log N$? Years ago Zeev Rudnick defined the Poisson generic real numbers as those whose binary expansions obey the Poisson law. Formally, a binary sequence x is Poisson generic if for all positive real numbers λ and for all non negative integers k,

$$\lim_{n\to\infty} \frac{\#\text{ length-}n\text{ words occur exactly }k\text{ times in first }[\lambda 2^n]\text{ symbols of }x}{\#\text{ length-}n\text{ words}} = e^{-\lambda}\frac{\lambda^k}{k!}.$$

Theorem (Yuval Peres and Benjamin Weiss, 2010).

(1) *Almost all (Lebesgue measure) real numbers are Poisson generic;*
(2) *All Poisson generic reals are Borel normal;*
(3) *The Champernowne number is not Poisson generic.*

No example of a particular instance of a Poisson generic real is known. Questions:

(1) Is there a computable Poisson generic real?
(2) Are all Martin-Löf random reals Poisson generic?
(3) Is it possible to characterize Poisson generic reals with some kind of Kolmogorov complexity?
(4) Prove that the set of Poisson generic reals is Π^0_3-complete.

The following talk by Weiss is relevant:
`https://www.youtube.com/watch?v=8AB7591De68`

(Chi Tat Chong) The tree coloring principle TT^1 states that every finite coloring of the full binary tree has an isomorphic monochromatic subtree. It is immediate that over RCA_0, $\mathsf{TT}^1 \not\to \mathsf{RT}^2_2$ since any ω-model of RCA_0 satisfies TT^1. However, in terms of inductive strength, TT^1 and RT^2_2 both imply Σ_2-bounding but not Σ_2-induction. Furthermore under $\neg I\Sigma_2$, every model of TT^1 has a recursive instance with no solution $\leq_T \emptyset''$. The same applies to RT^2_2.

Yet another similarity is exhibited in conservation: A recent result of Chong, Wang and Yang (in preparation), mirroring an earlier theorem of Patey and Yokoyama (2018) for RT^2_2, says that $\mathsf{TT}^1 + \mathsf{WKL}_0$ is Π^0_3-conservative over RCA_0. On the other hand, TT^1 is Π^1_1-conservative over $P\Sigma_1 + B\Sigma_2$, a property not known for RT^2_2.

Finally, over RCA_0, one has $\mathsf{RT}^2 \to \mathsf{TT}^1$, where $\mathsf{RT}^2 = \bigcup_k \mathsf{RT}^2_k$. These data points lead to the following question:

(1) Does $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ imply TT^1?

(Peter Hertling) If the so-called Hyperbolicity conjecture about the Mandelbrot set is true then the Mandelbrot set is computable in a strong sense. But this is currently unknown. It is even unknown whether the Mandelbrot set is a c.e. closed set. More details and further open computability-theoretic questions concerning the Mandelbrot set can be found in the article "Is the Mandelbrot set computable?", MLQ 51(1), 5–18, 2005, by P. Hertling.

(Steffen Lempp) I wanted to remind the participants of five long-standing open problems that the younger people may now be able to solve after all:

(1) Sacks (1963) asked if every locally countable partial order of size continuum embeds into the Turing degrees. He showed that the answer is positive under CH, and Abraham/Shore (1986) extended this by showing that any locally countable upper semilattice of size $\aleph_1$ embeds into the Turing degrees as an initial segment. Recently, Higuchi and Lutz (to appear) showed that every size-continuum poset $\mathcal{P}$ embeds into the Turing

degrees if $\mathcal{P}$ contains only chains of size at most 2, but that under AD, this may fail for $\mathcal{P}$ with chains of size at most 3.

(2) Lachlan and others (late 1960's) asked which finite lattices can be embedded into the c.e. Turing degrees. The best known result is a Π^0_2-condition for finite join-semidistributive lattices by Lerman (2000).
(3) Based on Rogers' homogeneity conjecture, the existence of nontrivial automorphisms of degree structures was raised. It has been completely solved for the m- and the c.e. m-degrees (Ershov/Palyutin 1975 and Dëgtev 1978) in the positive, and for the hyperarithmetic degrees (Slaman/Woodin/ 1990's) in the negative, but for the Turing and the enumeration degrees, we only know that there are at most countably many (Slaman/Woodin 1990's and Slaman/Soskova 2017).
(4) Ershov (1977) asked for which finite families F_1 and F_2 of c.e. sets their Rogers semilattices are isomorphic. Ershov (2003) showed that $(F_1', \subset) \cong (F_2', \subset)$ is a necessary condition, where F' is the finite partial order F with the maximal elements removed; it is conjectured that this condition is also sufficient.
(5) Muchnik/Semënov/Uspensky (1998) asked if Martin-Löf and Kolmogorov-Loveland randomness coincide, where the latter is defined in terms of computable non-monotonic adaptive martingales. Kastermans/Lempp (2010) were only able to separate Martin-Löf randomness from the nonadaptive version of KL randomness called injective randomness.

(Elvira Mayordomo) Classical Hausdorff (denoted $\dim_{\mathrm{H}}$) and packing dimension (denoted $\dim_{\mathrm{P}}$) are defined in terms of covers. Lutz and Lutz (2018) proved their point to set principles characterizing both Hausdorff and packing dimension for the space of reals in terms of their corresponding relativized effective dimensions. Using these principles, questions on classical dimensions can be rephrased as questions on effective dimensions. Several new results on classical fractal dimensions have already been proven in this fashion; see Lutz, Mayordomo (2021, to appear in the *Handbook of Computability and Complexity in Analysis*) and Lutz and Lutz (2020, *Complexity and Approximation*). We propose the following open questions:

(1) A set A is regular if $\dim_{\mathrm{H}}(A) = \dim_{\mathrm{P}}(A)$. Can computability (partially) characterize regularity for $A \subseteq 2^\omega$?
(2) It follows from work of Besicovitch (1952), Davis (1952), and Joyce, Preiss (1995) that for every $A \subseteq 2^\omega$ which is regular and analytic and every $s \leq \dim_{\mathrm{H}}(A)$, there is a closed subset $C \subseteq A$ with $s \leq \dim_{\mathrm{H}}(C)$. Can we give an alternative proof of this result using the point-to-set principles? Can we remove the analyticity requirement for regular sets?

For further motivation, please refer to Slaman's abstract in the present report.

(Arno Pauly) The Weihrauch lattice (see Brattka, Gherardi, Pauly (2017) for a survey) has a rich algebraic structure, but besides it forming a distributive lattice and a Kleene algebra, it does not seem to fit the mould of well-studied algebraic structures well. In particular, while we would expect the Weihrauch lattice to be the prototypic algebraic model of some kind of logic, it does not belong to the usual substructural logics studied (Brattka, Pauly (2018)). A potential next step would be to characterize further operations in an algebraic way:

(1) Can we characterize any further fragments of the theory of $(\mathfrak{W}; 0, 1, \emptyset, \sqcap, \sqcup, \times, \star, \rightarrow, ^*, ^\diamond, ^\frown)$ in a similar way?

A related group of issues is whether some of the operations are definable from others. It is clear that $0, \sqcap, \sqcup, \emptyset$ are definable from $\leq_W$ alone; and that $1, ^\star$ are definable from $\leq_W, \times$. It was recently shown by Westrick (2021) that $^\diamond$ is definable from $\leq_W, \star$. One particularly interesting question would be:

(2) Is 1 definable from $\leq_W$ alone?

An overview of recent development in the area, and further open questions, can be found in Pauly (An update on Weihrauch complexity, and some open questions, 2020).

(Vadim Puzarenko) We work within KPU, i.e., axioms of Kripke–Platek with Urelements. A wellfounded structure of KPU is called an admissible structure. We consider hereditarily finite structures (e.g., $\mathbb{HF}(\mathfrak{M})$) and hyper structures (e.g., $\mathbb{HYP}(\mathfrak{M})$) as main examples of admissibles. Under $\mathbb{A}$-c.e. and $\mathbb{A}$-c. sets we understand respectively Σ and Δ predicates on $\mathbb{A}$. All the considered structures are in some finite signature. Hence, such admissibles have universal c.e. predicates. Our question concerns interconnections between Computability Approaches on Admissibles and fixed points under the Jump operator.

There are two notions of relative computability among admissibles. First, we say that $\mathfrak{M}$ is Σ-*definable* in $\mathbb{A}$ iff there exists a map ν from $\mathbb{A}$ onto $\mathfrak{M}$ such that the ν-preimage of each signature relation on $\mathfrak{M}$ including equality is $\mathbb{A}$-c. Notice that $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\varnothing)$ iff $\mathfrak{M}$ is computable. Notion of $\mathfrak{M}$ to be Σ-definable in $\mathbb{HF}(A)$ corresponds to effective interpretability of $\mathfrak{M}$ in $\mathfrak{A}$ introduced in Harrison-Trainor, Melnikov, Miller, Montalbán (2017).

We give now a strong version of a computability level. We say that $\mathbb{A}$ is Σ-*reducible to* $\mathbb{B}$ (shortly, $\mathbb{A} \sqsubseteq_\Sigma \mathbb{B}$) iff there exists a map ν from $\mathbb{B}$ onto $\mathbb{A}$ such that all the ν-preimages of n-ary $\mathbb{A}$-c.e. relation are $\mathbb{B}$-c.e., for each natural number n. We say that $\mathbb{A}$ and $\mathbb{B}$ are Σ-*equivalent* (and denote it as $\mathbb{A} \equiv_\Sigma \mathbb{B}$) iff $\mathbb{A} \sqsubseteq_\Sigma \mathbb{B}$ and $\mathbb{B} \sqsubseteq_\Sigma \mathbb{A}$. Notice that $\mathfrak{M}$ is Σ-definable in $\mathbb{A}$ iff $\mathbb{HF}(\mathfrak{M}) \sqsubseteq_\Sigma \mathbb{A}$. As in classical case, each admissible set $\mathbb{A}$ has a directed graph $\mathfrak{M}_\mathbb{A}$ that $\mathbb{A} \equiv_\Sigma \mathbb{HF}(\mathfrak{M}_\mathbb{A})$.

Next we define the notion of the jump of a structure: $\mathcal{J}(\mathbb{A}) = (\mathbb{HF}(\mathfrak{M}_\mathbb{A}), P)$ is called a *jump structure of* $\mathbb{A}$ where P is a universal $\mathbb{A}$-c.e. relation. In comparison with classical case, the jump operation on structures has a fixed point as proved by Puzarenko (2011) and Montalbán (2013).

We conjecture that for every admissible set $\mathbb{A}$, the following conditions are equivalent:

(1) $\mathbb{A} \equiv_\Sigma \mathcal{J}(\mathbb{A})$;
(2) Σ-definability of $\mathbb{B}$ (as a structure) in $\mathbb{A}$ implies $\mathbb{B} \sqsubseteq_\Sigma \mathbb{A}$.

(1) $\Rightarrow$ (2) can be easily checked. As for (2) $\Rightarrow$ (1), at present, there are some examples which confirm the conjecture. See Puzarenko (2005) and Avdeev, Puzarenko (2019).

(Mariya Soskova) A set $A \subseteq \omega$ is *enumeration reducible* to a set $B \subseteq \omega$ (and we write $A \leq_e B$) if there is a c.e. set W such that $x \in A \Leftrightarrow \exists v[(x,v) \in W \,\&\, D_v \subseteq B]$. Let $W_e(A)$ denote the set which is enumeration reducible to A via W_e. Let $K_A = \bigoplus_{e<\omega} W_e(A)$. We define the *skip of* A to be $A^\diamond = \overline{K_A}$. The skip was introduced and investigated by Andrews, Ganchev, Kuyper, Lempp, Miller, Soskova, and Soskova (2019). It has the following properties:

1. (1) $A \leq_e B$ if and only if $A^\diamond \leq_1 B^\diamond$. (We can thus define the skip operator on degrees: $d_e(A)^\diamond = d_e(A^\diamond)$.)
2. (2) For every $A \geq_e \emptyset^\diamond$ there is some B such that $B^\diamond \equiv_e A^\diamond$.
3. (3) It is not always the case that $A \leq_e A^\diamond$. In fact, $A \leq_e A^\diamond$ if and only if A has cototal enumeration degree. The enumeration jump of a set A is defined as $A' = A \oplus A^\diamond$. So for cototal (and hence total) degrees $\mathbf{a}$ we have that $\mathbf{a}' = \mathbf{a}^\diamond$.
4. (4) $A \leq_e (A^\diamond)^\diamond$.

Questions:

1. (1) If G is arithmetically generic then for every n we have that the n-th skip of G is not below the $n+1$-st skip of G. We say that G exhibits a *zig-zag* behavior. Is there an arithmetical degree $\mathbf{a}$ such that for every n, the n-th skip of $\mathbf{a}$ is not below the $n+1$-st skip of $\mathbf{a}$? Equivalently, is there a degree $\mathbf{a}$ such that for every n the n-th skip of $\mathbf{a}$ is nontotal?
2. (2) There is a set A such that $(A^\diamond)^\diamond = A$. We call A a *skip* 2*-cycle*. AGKLMSS (2019) proved that each skip 2-cycle bounds every HYP degree. Can the HYP degrees be characterized as the degrees bounded by all skip 2-cycles? (Jun Le Goh provided an affirmative answer to (2) during the workshop: The set $\mathcal{P} = \{A : A \text{ is a skip 2-cycle}\}$ is a nonempty Σ^1_1 class. If $X \leq_e A$ for every $A \in \mathcal{P}$ then both X and $\overline{X}$ are Σ^1_1 in every member of a nonempty Σ^1_1 class. By the basis theorem of Harrington, Shore, Slaman (2017) X and $\overline{X}$ are Σ^1_1, hence X is HYP.)
3. (3) Is the skip operator first order definable in the enumeration degrees? An affirmative answer would yield a first order definition of the cototal degrees and by the above of the HYP degrees. (The HYP degrees are known to be definable through Slaman and Woodin's coding machinery though not with a simple definition).

(Frank Stephan) Recall the following notions: a set A is truth-table reducible to B iff there are recursive functions f, g such that $A(x) = f(x, B(0)B(1)\ldots B(g(x)))$ where $B(0)B(1)\ldots B(g(x))$ is a natural number representing the corresponding binary string of length $g(x)+1$. A is positive truth-table reducible to B iff for the same f, g it holds that whenever C is truth-table reducible to D and E to F via these f, g and $D \subseteq F$ then $C \subseteq E$. Frank Stephan (On the structure inside truth-table degrees, JSL 66(2), pages 731-770, 2001) showed that the possible numbers of positive degrees inside nonrecursive truth-table degrees are either odd or infinite, several odd numbers are found like 3, 19, 219, ..., namely all numbers m_n where m_n is the number of partial orders on a set of n elements. The number 1 only occurs in the recursive truth-table degree.

(1) What about the other odd numbers, can they occur as numbers of positive degrees inside truth-table degrees?

Furthermore, a set A is many-one reducible to B iff there is a recursive function f such that $A(x) = B(f(x))$ and A is one-one reducible to B if it is many-one reducible to B via a one-one function. These reductions induce the corresponding degrees. The following two questions were open 20 years ago and Frank Stephan has not seen anywhere since then the solutions to them, though he did not actively track the questions: Aleksandr Nikolaevich Dëgtev (tt- and m-degrees, Algebra and logic 12(2):78-89, 1973) showed that every r.e. many-one degree either consists of one one-one degree or is recursive or contains an infinite antichain of one-one degrees, but left open the question whether this holds for all many-one degrees; furthermore, Dëgtev (Partially ordered sets of 1-degrees, contained in recursively enumerable m-degrees. Algebra and Logic, 15(3):153-164, 1976) showed for every natural number n there is an r.e. many-one degree having one least degree and above it n minimal one-one degrees.

(2) Can there be infinitely many minimal one-one degrees in an r.e. many-one degree?

(Manlio Valenti) There are several ways the open and clopen Ramsey theorems can be phrased as computational problems. The uniform strength of the corresponding problems (as measured by Weihrauch reducibility $\leq_W$) has been explored in Marcone, Valenti (to appear). In particular, we can consider the problem $\mathbf{\Sigma}^0_1$$-$RT of finding a homogeneous solution for an open $P \subset [\mathbb{N}]^{\mathbb{N}}$, and its restriction $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$ to problems that have no homogeneous solutions that land in the set. Can we use these problems to uniformly solve the problem $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ of computing a path through an ill-founded subtree of $\mathbb{N}^{<\mathbb{N}}$? Namely:

(1) $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_W \mathbf{\Sigma}^0_1\text{-}\mathsf{RT}$?
(2) $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_W \mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$?

Separately, in Goh, Pauly, Valenti (to appear), we explored the uniform computational strength of the problem DS of finding a descending sequence through an ill-founded linear order. A question that resisted full characterization is:

(3) $\mathsf{KL} \leq_W \mathsf{DS}$?

Among the possible strengthenings of DS, we can consider the problem $\mathbf{\Gamma}$-DS, where the input order is represented via a $\mathbf{\Gamma}$-code, with $\mathbf{\Gamma}$ among $\mathbf{\Sigma}^0_k$, $\mathbf{\Pi}^0_k$, $\mathbf{\Delta}^0_k$, $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, $\mathbf{\Delta}^1_1$. We can also consider the problem $\mathbf{\Gamma}$-BS, where we ask for a bad sequence through a non-well quasi-order, and its restrictions $\mathbf{\Gamma}\text{-}\mathsf{BS}_{LQO}$ and $\mathbf{\Gamma}\text{-}\mathsf{BS}_{PO}$ respectively to total quasi-orders and to partial orders. We showed that $\mathbf{\Sigma}^0_1\text{-}\mathsf{DS} <_W \mathbf{\Sigma}^0_1\text{-}\mathsf{BS}_{LQO}$.

(4) What are the exact relations between $\mathbf{\Sigma}^0_1$-DS, $\mathbf{\Sigma}^0_1$-BS_{LQO}, $\mathbf{\Sigma}^0_1$-BS_{PO}, $\mathbf{\Sigma}^0_1$-BS, and $\mathbf{\Delta}^0_2$-DS?

(Liang Yu) *Turing determinacy* (TD) says that for every cofinal set A of *Turing degrees*, A contains an upper cone. *Strong Turing determinacy* (sTD) says that for every set A of *reals* with cofinal range in the Turing degrees, A has a pointed subset (a pointed set is a perfect set in which every real computes a representation of the set). Over ZF,

(1) Does TD imply $\mathrm{DC}_{\mathbb{R}}$?
(2) Does TD(+DC) imply sTD?

(Mars Yamaleev) A positive answer for the following question could be the essential step for the long-standing problem of definability of c.e. Turing degrees in the partial ordering of 2-c.e. Turing degrees.

(1) Given a 2-c.e. set D of properly 2-c.e. Turing degree, does there exist a c.e. set A such that $D \not\leq_T A$ and for any c.e. $W \leq_T D$ we have $W \leq_T A$?

Second, Soare and Stob proved in 1982 that for any noncomputable low c.e. set C there exists a set D which is $CEA(C)$ and not of c.e. Turing degree. Arslanov in 2011 showed that if this C is superlow then the degree of D must be 2-c.e.

(2) Can this property characterize superlow c.e. degrees? In other words, given a noncomputable low, but not superlow, c.e. set C, does there exist a set D which is $CEA(C)$ and not of 2-c.e. Turing degree?

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