Bi-embeddable Categoricity of Computable Structures

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Bi-embeddable structures

In the classical computable structure theory, one typically considers algorithmic properties of the *isomorphism type* of a structure $\mathcal{S}$.

In this talk, we work with *bi-embeddability types*.

Two structures $\mathcal{A}$ and $\mathcal{B}$ are **bi-embeddable** (or equimorphic), denoted by $\mathcal{A} \approx \mathcal{B}$, if there exist isomorphic embeddings

$$f\colon \mathcal{A} \hookrightarrow \mathcal{B} \text{ and } g\colon \mathcal{B} \hookrightarrow \mathcal{A}.$$

Known results on bi-embeddable structures

Some of the first computability-theoretic results on bi-embeddability types were obtained by Montalbán (2005), and Greenberg and Montalbán (2008).

Theorem

Let $\mathcal{S}$ be a hyperarithmetical structure from one of the classes given below. Then there is a computable structure $\mathcal{A}$ such that $\mathcal{A} \approx \mathcal{S}$.

- linear orders; [Montalbán 2005]
- Boolean algebras; [Greenberg and Montalbán 2008]
- abelian p-groups. [Greenberg and Montalbán 2008]

Known results on bi-embeddable structures

Fokina, Rossegger, and San Mauro (2019) started investigations of degree spectra up to bi-embeddability.

For a countably infinite structure $\mathcal{S}$, the *bi-embeddability spectrum* of $\mathcal{S}$ is the set

$$\mathrm{DgSp}_{\approx}(\mathcal{S}) = \{\deg(\mathcal{A}) : \mathcal{A} \approx \mathcal{S} \text{ and } \mathrm{dom}(\mathcal{A}) = \omega\}.$$

A lot of known examples of classical degree spectra of structures can be realized as bi-embeddability spectra.

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A lot of known examples of classical degree spectra of structures can be realized as bi-embeddability spectra.

The following question is still open:

Problem (Fokina, Rossegger, and San Mauro 2019)

Is there a bi-embeddability spectrum, which is not a (classical) degree spectrum of a structure? What about vice versa?

Categoricity in the bi-embeddability setting

Definition (Mal'tsev 1961)

A computable structure $\mathcal{S}$ is **computably categorical** (or autostable) if for any computable isomorphic copy $\mathcal{A}$ of $\mathcal{S}$, there is a computable isomorphism $f\colon \mathcal{A} \to \mathcal{S}$.

Definition

A computable structure $\mathcal{S}$ is **computably bi-embeddably categorical** (or *computably b.e. categorical*, for short) if for any computable structure $\mathcal{A}$ bi-embeddable with $\mathcal{S}$, there are computable isomorphic embeddings $f\colon \mathcal{A} \hookrightarrow \mathcal{S}$ and $g\colon \mathcal{S} \hookrightarrow \mathcal{A}$.

The definitions above are relativized in a natural way:
For a Turing degree $\mathbf{d}$, one obtains the notions of $\mathbf{d}$-computable categoricity and $\mathbf{d}$-computable b.e. categoricity.

(i) Bi-embeddable categoricity spectra.
(ii) Index sets.
(iii) Bi-embeddable categoricity for familiar classes of structures.

Bi-embeddable categoricity spectra

The **categoricity spectrum** of a computable structure $\mathcal{S}$ is the set

$$\mathrm{CatSpec}(\mathcal{S}) = \{\mathbf{d} : \mathcal{S} \text{ is } \mathbf{d}\text{-computably categorical}\}.$$

Similarly, one defines the **bi-embeddable categoricity spectrum** for $\mathcal{S}$:

$$\mathrm{CatSpec}_{\approx}(\mathcal{S}) = \{\mathbf{d} : \mathcal{S} \text{ is } \mathbf{d}\text{-computably b.e.-categorical}\}.$$

The least degree, if it exists, in $\mathrm{CatSpec}(\mathcal{S})$ (in $\mathrm{CatSpec}_{\approx}(\mathcal{S})$) is called the *degree of categoricity* for $\mathcal{S}$ (the *degree of bi-embeddable categoricity* for $\mathcal{S}$, respectively).

Bi-embeddable categoricity spectra

There are a lot of known examples of categoricity spectra: e.g.,

Theorem (Fokina, Kalimullin, and Miller 2010; Csima, Franklin, and Shore 2013)

Let α be a computable non-limit ordinal. Then any Turing degree $\mathbf{d}$, which is d.c.e. in and above $\mathbf{0}^{(\alpha)}$, is a degree of categoricity.

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Some of these examples can be transferred into the bi-embeddability setting:

Theorem 1 (B., Fokina, Rossegger, and San Mauro 2021)

Let α be a computable non-limit ordinal.

(a) Any degree $\mathbf{d}$, which is d.c.e. in and above $\mathbf{0}^{(\alpha)}$, is a degree of bi-embeddable categoricity.

(b) The set of PA degrees over $\mathbf{0}^{(\alpha)}$ is a bi-embeddable categoricity spectrum.

Theorem 1: Using bi-embeddable triviality

The key notion employed in the proof of Theorem 1 is that of *bi-embeddable triviality*.

A structure $\mathcal{S}$ is *bi-embeddably trivial* (or b.e. trivial, for short) if any structure $\mathcal{A}$, which is bi-embeddable with $\mathcal{S}$, is isomorphic to $\mathcal{S}$.

Roughly speaking, our proof combines the following:

- The pairs of structures technique by Ash and Knight $\rightsquigarrow$ If one works with pairs of ordinals, then the b.e. triviality of the resulting structure $\mathcal{S}$ is almost immediate.
- Known techniques for categoricity spectra:
  - the construction for d.c.e. degree of categoricity, by Fokina, Kalimullin, and Miller (2010);
  - the construction for categoricity spectrum containing precisely PA degrees [essentially, Miller 2009]. $\square$

Degrees of b.e. categoricity, revisited

It turns out that there is a *much easier* way to obtain further examples of degrees of b.e. categoricity.

Recall that a total function $f\colon \omega \to \omega$ is a Π^0_1 *function singleton* if there is a computable tree $T \subseteq \omega^{<\omega}$ such that f is the unique path through T.

Proposition 1

Every degree $\mathbf{d} \geq \mathbf{0}'$, which contains a Π^0_1 function singleton, is a degree of bi-embeddable categoricity.

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Every degree $\mathbf{d} \geq \mathbf{0}'$, which contains a Π^0_1 function singleton, is a degree of bi-embeddable categoricity.

Corollary

Let α be a non-zero computable ordinal. If $\mathbf{0}^{(\alpha)} \leq \mathbf{d} \leq \mathbf{0}^{(\alpha+1)}$, then $\mathbf{d}$ is a degree of bi-embeddable categoricity.

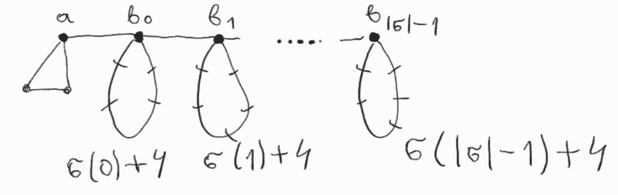
To our best knowledge, it is still open whether an analogue of this corollary holds for the case of isomorphisms.

Note that Csima and Ng announced that every Δ^0_2 degree is a degree of categoricity.

Proof sketch for Proposition 1

Recall that an undirected graph G is *strongly locally finite* (or a slf-graph, for short) if each component of G is finite.

For a string $\sigma \in \omega^{<\omega}$, we define a finite graph H_σ:



It is clear that $H_\sigma \hookrightarrow H_\tau$ if and only if $\sigma \subseteq \tau$.

For a tree $T \subseteq \omega^{<\omega}$, the slf-graph $\underline{G}(T)$ is defined as disjoint union of all H_σ, where $\sigma \in T$.

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For a tree $T \subseteq \omega^{<\omega}$, the slf-graph $\underline{G}(T)$ is defined as disjoint union of all H_σ, where $\sigma \in T$.

Let f be a Π^0_1 function singleton. Let T be a computable tree, which witnesses this fact. Then the graph $G = \underline{G}(T)$ has degree of b.e. categoricity $\deg_T(f)$.

Key Observation: If a graph $\mathcal{A}$ is bi-embeddable with G, then $\mathcal{A}$ is disjoint union of the following components:

- H_σ, for each $\sigma \in T$ such that $\sigma \not\subset f$.
- Graphs S such that $H_{\sigma_S} \hookrightarrow S \hookrightarrow H_{\tau_S}$ for some $\sigma_S \subseteq \tau_S \subset f$. In addition, there are infinitely many such S.

□

Open Problem

Is there a bi-embeddable categoricity spectrum, which is not a categoricity spectrum? What about vice versa?

(II) The complexity of index sets

$\mathbf{0}'$-computable b.e. categoricity

Downey, Kach, Lempp, Lewis-Pye, Montalbán, and Turetsky (2015) proved that the index set of computably categorical structures is Π^1_1-complete.

Within the bi-embeddability framework, it is not hard to obtain the following result:

Theorem 2 (B., Fokina, Rossegger, and San Mauro 2018)

The index set of $\mathbf{0}'$-computably bi-embeddably categorical, strongly locally finite graphs is Π^1_1-complete.

Proof of Theorem 2

(1) Choose a computable sequence of trees $(T_k)_{k\in\omega}$ such that

- if $k \in \mathcal{O}$, then T_k is well-founded;
- if $k \notin \mathcal{O}$, then T_k is ill-founded and T_k has no hyperarithmetical paths.

(2) We consider a computable sequence $(\underline{G}(T_k))_{k\in\omega}$.

- If $k \in \mathcal{O}$, then T_k is well-founded. This implies that $\underline{G}(T_k)$ is bi-embeddably trivial.
  Since $\underline{G}(T_k)$ is $\mathbf{0}'$-computably categorical, $\underline{G}(T_k)$ is also $\mathbf{0}'$-computably b.e. categorical.
- If $k \notin \mathcal{O}$, then consider two structures

$$G = \underline{G}(T_k) \text{ and } G_1 = \underline{G}(T_k) \sqcup H_\Lambda,$$

  where Λ is the empty string. The graphs G and G_1 are bi-embeddable.
  Every embedding $f\colon G_1 \hookrightarrow G$ computes a path through T_k.
  Hence, $\underline{G}(T_k)$ is not hyperarithmetically b.e. categorical. □

Computable b.e. categoricity

The following question was open:

Problem

Find the complexity of the index set for *computably* bi-embeddably categorical structures.

We answer this question:

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We answer this question:

Theorem 3

The index set of computably bi-embeddably categorical structures is Π^1_1-complete.

Proof sketch for Theorem 3

This is an "enhanced" version of Theorem 2.

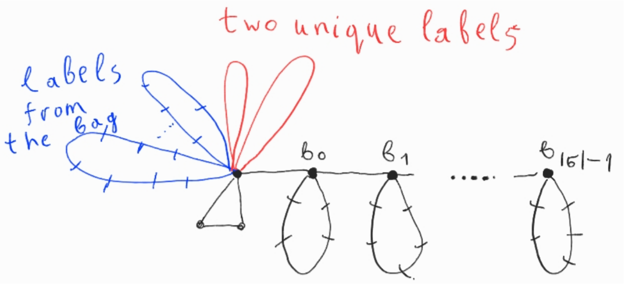
The key ingredient of the proof is the following construction.

Given a computable infinite tree $T \subseteq \omega^{<\omega}$, we produce a computable structure $\mathcal{S}(T)$ such that:

- $\mathcal{S}(T)$ is computably categorical;
- if T is well-founded, then $\mathcal{S}(T)$ is b.e. trivial;
- if T is ill-founded, then there is a computable structure $\mathcal{M} \approx \mathcal{S}(T)$ such that every embedding $f\colon \mathcal{M} \hookrightarrow \mathcal{S}(T)$ computes a path through T.

A modification of the technique of *pushing on isomorphisms*.

There will be a c.e. *bag of labels*, which is shared by all strategies.



Strategy τ for a string $\sigma \in T$ — It has only one outcome.

1. When first visited, τ adds its own copy of H_σ: The first vertex has all labels from the bag, and two additional unique labels (the *elder* one and the *younger* one).
2. Whenever τ is visited again, we refresh the labels:
   - Add all missing labels from the bag.
   - Enumerate the elder label into the bag. The younger laber becomes the elder one. Add a fresh younger label.

Let $(\mathcal{A}_e)_{e\in\omega}$ be the standard computable list of computable graphs. Denote $\mathcal{S} := \mathcal{S}(T)$.

Requirement P_e. If $\mathcal{A}_e \cong \mathcal{S}$, then there is a computable isomorphism f from $\mathcal{S}$ onto $\mathcal{A}_e$.

Strategy τ for P_e — Outcomes: $\infty < \dots < 2 < 1 < 0$.

When τ is visited, let k be the number of times τ has had outcome ∞.

We try to extend the isomorphism f for all components, which were added by the strategies ζ satisfying one of the following:

- ζ is incomparable with τ,
- $\zeta \supseteq \tau\widehat{\ }\mathtt{m}$ for some $m < k$;
- $\zeta \supseteq \tau\widehat{\ }\infty$.

If f is successfully extended, then τ has outcome ∞.
Otherwise, τ has outcome $\mathtt{k}$.

Verification Sketch.

Our structure $\mathcal{S}(T)$ is the disjoint union of:

- $\underline{G}(T)$, with all labels from the bag attached (this structure is built along the true path of the tree of strategies);
- an infinite family of finite graphs — each of these graphs has its own unique label.

(a) The requirements P_e ensure that $\mathcal{S}(T)$ is computably categorical.

(b) If T is well-founded, then the b.e. triviality of $\underline{G}(T)$ guarantees that $\mathcal{S}(T)$ is also b.e. trivial.

(c) If T is ill-founded, then consider

$$\mathcal{S}(T) \text{ and } \mathcal{S}(T) \sqcup (\text{the copy of } H_\Lambda \text{ with all bag labels attached}).$$

These structures are bi-embeddable. □

(III) Bi-embeddable categoricity for familiar classes

Boolean algebras

The bi-embeddability types of computable Boolean algebras $\mathcal{B}$ have a pretty simple classification:

- If $\mathcal{B}$ is not superatomic, then $\mathcal{B}$ is bi-embeddable with the atomless Boolean algebra.
  In this case, one can show that $\mathcal{B}$ is not hyperarithmetically b.e. categorical.
- If $\mathcal{B}$ is superatomic, then $\mathcal{B}$ is bi-embeddably trivial.

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- If $\mathcal{B}$ is not superatomic, then $\mathcal{B}$ is bi-embeddable with the atomless Boolean algebra.
  In this case, one can show that $\mathcal{B}$ is not hyperarithmetically b.e. categorical.
- If $\mathcal{B}$ is superatomic, then $\mathcal{B}$ is bi-embeddably trivial.

Theorem 4 (B., Rossegger, and Zubkov)

Let α be a non-zero computable ordinal, and let k be a non-zero natural number. The superatomic Boolean algebra $\mathrm{Int}(\omega^\alpha \cdot k)$ has degree of bi-embeddable categoricity

$$\begin{cases} \mathbf{0}^{(2\alpha-1)}, & \text{if } \alpha < \omega, \\ \mathbf{0}^{(2\alpha)}, & \text{if } \alpha \geq \omega. \end{cases}$$

Scattered linear orders of finite Hausdorff rank

Recall that the rank of a scattered linear order $\mathcal{L}$ can be defined as follows:

- $\mathbf{VD}_0 = \{0, 1\}$;
- $\mathbf{VD}_\alpha = \left\{ \sum_{i\in\tau} \mathcal{L}_i : \ \mathcal{L}_i \in \bigcup_{\beta<\alpha} \mathbf{VD}_\beta, \text{ and } \tau \in \{\omega, \omega^*, \zeta\} \right\}$, for an ordinal $\alpha > 0$.

The *Hausdorff rank* of $\mathcal{L}$ is the least α such that $\mathcal{L} \in \mathbf{VD}_\alpha$.

The VD^**-rank* of $\mathcal{L}$ is the least α such that $\mathcal{L}$ is a finite sum of orders from $\mathbf{VD}_\alpha$.

Theorem 5 (B., Rossegger, and Zubkov)

Let $\mathcal{L}$ be a computable linear order with VD^*-rank $n+1$. Then $\mathcal{L}$ is $\mathbf{0}^{(2n+1)}$-computably bi-embeddably categorical, but not $\mathbf{0}^{(2n)}$-computably bi-embeddably categorical.

References

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