Two Vignettes

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Oberwolfach

https://www.nd.edu/~cholak/papers/oberwolfach2021.pdf

Thanks and Apologies

Overall Theme

Anything which can happen in computability theory happens somewhere in the study of the c.e. sets and degrees. Perhaps really just fun with effective constructions.

The Collapse of an REA hierarchy

On work with Peter Hinman (1994), work with Peter Gerdes (not available yet), and work of Peter Gerdes (2020) plus a new question from Gerdes.

Fun with Peter^2

A is 1-**R**ecursively **E**numerable and **A**bove in X (1-REA in X) iff $A = X \oplus W_e^X$, for some e. W_e^X itself not need compute X.

A is $(n+1)$-REA in X iff A is 1-REA in Y and Y is n-REA in X.

A is n-REA iff it is n-REA in $\emptyset$. A set A has n-REA degree iff it is Turing equivalent to a n-REA set.

A 1-REA set is properly 1-REA iff it is not computable. A $(n+1)$-REA set is properly $(n+1)$-REA iff it is does not have n-REA degree.

1-REA Sets

$A = A^{[1]} = W_{e_1}$, for some e_1. What enters stays.

2-REA Sets

$A = A^{[1]} \sqcup A^{[2]}$ and $A^{[2]} = W_{e_2}^{A^{[1]}}$, for some e_2. Axioms cannot be reused.

3-REA Sets

$A = A^{[1]} \sqcup A^{[2]} \sqcup A^{[3]}$ and $A^{[3]} = W_{e_3}^{A^{[\leq 2]}}$, for some e_3.

The Results

Theorem (Soare and Stob 1982)

Every properly 1-REA set A can be nonuniformly extended to a properly $(1+1)$-REA set $A \oplus W_e^A$.

Theorem (Cholak and Hinman 1994)

Let m be a positive integer. Every properly 1-REA set A can be nonuniformly extended to a properly $(1+m)$-REA set. Every properly 2-REA set A can be nonuniformly extended to a properly $(2+m)$-REA set.

Theorem (Cholak and Gerdes)

There is a properly 3-REA set A which cannot be extended to a properly $(3+1)$-REA set.

The Extendability Results

The fact the extension must be nonuniform uses Jockusch and Shore's Hop Inversion (published in 1985) and the Recursion Theorem.

Given A properly 2-REA and let $m = 1$. Build two sets $U^A_{e_0}$ and $U^A_{e_1}$ such that, for all 2-REA sets X_e, we meet the following for all j, e, j' and e':

$$\mathcal{R}_{j,e,j',e'}\colon\ \Phi_j(A \oplus U^A_{e_0}) \neq X_e \text{ or } \Phi_j(X_e) \neq A \oplus U^A_{e_0}, \text{ or}$$
$$\Phi_{j'}(A \oplus U^A_{e_1}) \neq X_{e'} \text{ or } \Phi_{j'}(X_{e'}) \neq A \oplus U^A_{e_1}.$$

Uses the true stages approximation and finite injury.

The Requirements for the Nonextendability Result

Build 3-REA sets A and Y_i and Turing Functionals Γ_i and Θ such that, for all 2-REA sets X_e, we meet the following for all i, j, e:

$\mathcal{P}_i$: $\Gamma_i(A \oplus W_i^A) = Y_i$ and $\Theta(Y_i) = W_i^A$.

$\mathcal{R}_{j,e}$: $\Phi_j(A) \neq X_e$ or $\Phi_j(X_e) \neq A$.

Again uses the true stages approximation and finite injury.

ω-REA sets

$A^{[i]} = W_{f(i)}^{A^{[<i]}}$, where f is computable.

If there is a least i such that $A^{[i]}$ is not computable then A computes a non computable Σ_1^0 set. Otherwise A is computable in $\mathbf{0}''$ as the union of computable sets.

Theorem (Gerdes)

There is a ω-REA set A such that A and $0'$ form a mininal pair.

Question (Gerdes)

Is there a ω-REA set A where all $A^{[i]}$ are low$_2$ but A computes $0'''$?

Low$_2$

Theorem (Cholak, Downey, Greenberg 2022)

If A is low$_2$ then $\mathcal{L}(A)$ and $\mathcal{E}$ are isomorphic.

The issue is access to elements of $\overline{A}$.

Domination

Definition

Given two functions g and r from the nationals to the nationals, g *dominates* r iff, there is a k, for all $l \geq k$, $g(l) \geq r(l)$.

Theorem (Martin)

H is high iff $H' \equiv_T 0''$ iff there is a function g of Turing degree H which dominates all computable functions.

Corollary

A is low$_2$ iff $A'' \equiv_T 0''$ iff $0'$ is high over A ($(0')' \equiv_T A''$) iff there is function g of Turing degree $0'$ which dominates all A-computable functions.

Low$_2$ Access

Uniformly stagewise construct sets F_i, such that, for all i, $F_i \cap \overline{A}$ is nonempty. If $F_{i,s} \cap \overline{A}_s$ is empty add every ball outside A which is below some *large* ball into F_e.

Stagewise define $h_s^{A_s}(e)$ as the maximum element of $F_{i,s} \cap \overline{A}_s$ with same use.

We will e_k-certify the balls in F_e at stage $s+1$ if $g_{s+1}(e) \geq h_s^{A_s}(e)$, where it could be that g dominates h from k onward at stage s.

Since A is low$_2$, for some least k, for almost all e, the balls inside F_e will be e_k-certified. By the use of *largeness*, some of these balls will be *freshly* e_k-certified at the final certification stage for F_e.

For each possible k, consider the e_k-certified balls as elements of $\overline{A}$ and use them accordingly to construct what is needed but one such object for each possible k. The k makes this harder to iterate.

Time for a diagram?