Effectively Hausdorff Spaces

Matthias Schröder

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Survey

- The previous notion of Computable Hausdorffness
- A new definition of effective Hausdorffness
- Compact overt choice for effectively Hausdorff spaces
- A characterisation of computable multifunctions

Previous Definition
X is *computably Hausdorff*, if inequality on X is semi-decidable.

Characterisation (A. Pauly 2012)
Let X be an admissibly represented QCB_1-space. TFAE:

- X is computably Hausdorff.
- The diagonal $\{(x,x) \mid x \in X\}$ is co-c.e. closed.
- The embedding $X \hookrightarrow \mathcal{A}(X)$, $x \mapsto \{x\}$ is computable.
- The inclusion $\mathcal{K}(X) \hookrightarrow \mathcal{A}(X)$ is well-defined and computable.
- $\cap\colon \mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X)$ is well-defined and computable.

Question (A. Pauly, Oberwolfach Report 1/2018, Question 3)
Is any computably compact, computably Hausdorff space also computably regular?

Classical Theorem
Any compact Hausdorff space is regular.

Remember

- X is *computably compact*, if, for U open, '$U = X$?' is semi-decidable.
- X is *computably regular*, if, given $x \in U \in \mathcal{O}(X)$, one can computably select an open set V and a closed set A such that $x \in V \subseteq A \subseteq U$.

Answer: No!

Counterexample

Let $\omega\mathsf{M}$ be the one-point compactification of a computable metric space M that is *not* locally compact.

- $\omega\mathsf{M}$ is computably Hausdorff and computably compact.
- But $\omega\mathsf{M}$ is *not* topologically regular,
- hence not computably regular.

One-point compactification of M:

- Underlying set: $\omega\mathsf{M} := \mathsf{M} \cup \{\omega\}$
- Topology: $\mathcal{O}(\mathsf{M}) \cup \{\omega\mathsf{M} \setminus K \mid K \text{ compact in } \mathsf{M}\}$
- $\omega\mathsf{M}$ has a canonical representation $\delta_{\omega\mathsf{M}}$ derived from δ_{M}.

Main Problem

- Computable Hausdorffness $\not\Rightarrow$ topological Hausdorffness.
- By contrast:
  - Computable compactness $\Rightarrow$ topological compactness.
  - Computable regularity $\Rightarrow$ topological regularity.
- However:
  Computable Hausdorffness $\Rightarrow$ sequential Hausdorffness.

Remark

Sequentially Hausdorff: every convergent sequence has a unique limit.

A new notion of effective Hausdorffness

Why not the following definition?

Call X *effectively* T_2, if there are computable *separators* $\mathrm{U}, \mathrm{V} \colon \mathrm{X} \times \mathrm{X} \dashrightarrow \mathcal{O}(\mathrm{X})$ s.t.

$$x \neq y \Longrightarrow x \in \mathrm{U}(x,y),\ y \in \mathrm{V}(x,y),\ \mathrm{U}(x,y) \cap \mathrm{V}(x,y) = \emptyset\,.$$

Example

Any computable metric space has computable separators:

$$\mathrm{U}(x,y) := B_d\big(x, \tfrac{d(x,y)}{2}\big),\ \ \mathrm{V}(x,y) := B_d\big(y, \tfrac{d(x,y)}{2}\big)\,.$$

Disadvantage

- U, V do not provide any *finite* separation information, because any prefix of a standard name of an open set can be extended to a name of the open set X.
- Semi-decidability of inequality is not implied.

Counterexample

Define X by

- $X := \{2a \mid a \in \mathbb{N}\} \cup \{2a+1 \mid a \in \mathbb{N} \setminus \mathrm{H}\}$,
  where H is the Halting-Problem.
- $\delta_X(2a\,0^\omega) := 2a$,
  $\delta_X(2a{+}1\,0^\omega) := \begin{cases} 2a+1 & \text{if } a \notin \mathrm{H} \\ 2a & \text{if } a \in \mathrm{H} \end{cases}$.

Then

- U, V defined by
  $$\mathrm{U}(x, y) := \{x\},\ \mathrm{V}(x, y) := \{y\}$$
  are computable separators for X.
- But inequality on X is not semi-decidable.

Basic facts from Computable Analysis / TTE

- Basic objects: represented spaces $\mathsf{X} = (X, \delta_{\mathsf{X}})$.
- QCB $=$ class of top. spaces that can be handled by TTE.
- *QCB-space*: a **q**uotient of a **c**ountably **b**ased top. space.
- Example: the final topology of a TTE-representation is QCB.
- *Effective QCB-space*: a represented space $\mathsf{X} = (X, \delta_{\mathsf{X}})$ s.t. δ_{X} is computably admissible.
- Effective QCB-spaces have excellent closure properties:
  - cartesian closed
  - finite limits
  - finite colimits
- From δ_{X} one derives computably admissible representations:
  - θ_+ for the open subsets of X
  - ψ_- for the closed subsets of X
  - κ_- for the compact subsets K of X, providing information about open sets containing K
  - κ_+ for the non-empty compact subsets K of X, providing information about open sets intersecting K

Basic Idea

Proposition

Let X be a Hausdorff QCB-space.

- X has a subtopology $\tau \subseteq \mathrm{O}(X)$ that
  - has a countable base and is Hausdorff.
- Any such subtopology τ satisfies:
  - $\tau|_K = \mathrm{O}(X)|_K$ for any compact subspace $K \in \mathrm{K}(X)$.
  - $(x_n)_n$ converges to x_∞ in X iff
    (a) $(x_n)_n$ converges to x_∞ wrt. τ &
    (b) $(x_n)_n$ is contained in some $K \in \mathrm{K}(X)$.

Definition

A *computable witness of Hausdorffness* for X is a sequence $(\mathrm{u}_i, \mathrm{v}_i)_i$ in $\mathcal{O}(\mathsf{X}) \times \mathcal{O}(\mathsf{X})$ such that:

- $\mathrm{u}_i \cap \mathrm{v}_i = \emptyset$ for all $i \in \mathbb{N}$.
- For all $x \neq y$, there is some i such that $x \in \mathrm{u}_i$, $y \in \mathrm{v}_i$.
- The maps $i \mapsto \mathrm{u}_i$, $i \mapsto \mathrm{v}_i$ are computable wrt. θ_+.
- It is called *strong*, if additionally
  - $\{\mathrm{u}_j, \mathrm{v}_j \mid j \in \mathbb{N}\}$ is an effective base of some topology τ.
  - For (i, j) one can compute k s.t. $(\mathrm{u}_k, \mathrm{v}_k) = (\mathrm{u}_i \cap \mathrm{u}_j, \mathrm{v}_i \cup \mathrm{v}_j)$.

Definition

A represented space X is an *effectively Hausdorff QCB-space*, if

- it has a computable witness of Hausdorffness &
- its representation δ_{X} is computably admissible.

Example (Effectively Hausdorff QCB-space)
Any computable metric space.

Theorem
Let X and Y be effectively Hausdorff QCB-spaces.

- X is topologically Hausdorff.
- Inequality on X is semi-decidable,
  hence X is computably Hausdorff according to the previous definition.
- $X \times Y$ and $X \oplus Y$ are effectively Hausdorff.
- Any QCB-subspace of X is effectively Hausdorff.
- If Z has a computable dense sequence, then Y^Z is an effectively Hausdorff QCB-space.

Reformulation of A. Pauly's Question

Is any computably compact, *effectively Hausdorff QCB-space* also computably regular?

Answer: Yes.

Theorem

Let X be a computably compact, effectively Hausdorff QCB-space.

- X is computably regular.
- X has an effective countable base.
- If X has a computable dense sequence $(\alpha_k)_k$, then X has a metric d such that
  - (X, d, α) is a computable metric space &
  - its Cauchy representation is computably equivalent to δ_X.

Compact overt choice

Compact overt choice

- Selecting an element in a compact subset given by positive information,
- i.e., the computational problem $\mathrm{KVC}_{\mathsf{Y}} \colon \mathrm{K}_{+}(\mathsf{Y}) \rightrightarrows \mathsf{Y}$,

  $$\mathrm{KVC}_{\mathsf{Y}}[K] := \{y \mid y \in K\} \quad \text{for } K \in \mathrm{K}_{+}(\mathsf{Y}) := \mathrm{K}(\mathsf{Y}) \setminus \{\emptyset\}$$

  where
  - Y is a represented QCB-space,
  - $\mathrm{K}_{+}(\mathsf{Y})$ carries a *positive* representation like κ_{+}.

Proposition

Compact overt choice is computable wrt. κ_+ for:

- [V. Brattka & P. Hertling 1994]
  any computable metric space;
- [M. de Brecht & A. Pauly & Sch. 2019]
  any computably Hausdorff, computable quasi-Polish space.

Proposition

Let $\mathsf{Y} \in \mathsf{QCB}_2 \setminus \omega\mathsf{Top}$.

- Compact overt choice $\mathrm{KVC}_{\mathsf{Y}}$ for Y is incomputable wrt. κ_+.
- $\mathrm{ACC}_{\mathbb{N}} \leq_{\mathrm{W}}^{\mathrm{top}} \mathrm{KVC}_{\mathsf{Y}}$.

Remark

- $\mathrm{ACC}_{\mathbb{N}}$: the problem *all-or-co-unique choice* for $\mathbb{N}$
- $\leq_{\mathrm{W}}^{\mathrm{top}}$: the topological version of Weihrauch reducibility

How can we turn $\mathrm{KVC}_{\mathsf{Y}}$ computable for $\mathsf{Y} \notin \omega\mathsf{Top}$?

Idea: Use a more informative representation for $\mathrm{K}_+(\mathsf{Y})$.

Definition

Let Y be an effectively Hausdorff QCB-space.

- Define a representation $\kappa_{+\mathrm{b}}$ for $\mathrm{K}_+(\mathsf{Y})$ by

$$\kappa_{+\mathrm{b}}\langle p, b\rangle = K \quad \text{iff} \quad \kappa_+(p) = K \;\&\; K \subseteq \kappa_-(b)$$

  where κ_+, κ_- are the positive / negative representations for $\mathrm{K}_+(\mathsf{Y})$.
- Define:
  - $\mathcal{K}_{+\mathrm{b}}(\mathsf{Y}) := (\mathrm{K}_+(\mathsf{Y}), \kappa_{+\mathrm{b}})$
  - $\mathcal{K}_{+}(\mathsf{Y}) := (\mathrm{K}_+(\mathsf{Y}), \kappa_{+})$
  - $\mathcal{K}_{-}(\mathsf{Y}) := (\mathrm{K}(\mathsf{Y}), \kappa_{-})$

Remark

- $\mathcal{K}_{+\mathrm{b}}(\mathsf{Y})$ is topological iff Y is compact.
- $\mathcal{K}_{+\mathrm{b}}(\mathsf{Y})$ has the convergence relation of a filter space.

Theorem
Let Y be an effectively Hausdorff QCB-space.

- Compact overt choice for Y is computable wrt. $\kappa_{+\mathrm{b}}$,
- i.e., there is a computable selector $S\colon \mathrm{dom}(\kappa_{+\mathrm{b}}) \to Y$ such that $S(p) \in \kappa_{+\mathrm{b}}(p)$.

Characterising computable multifunctions

Recap

- A *multifunction* *(or computational problem)* F is a relation between represented spaces X, Y, written as $F\colon \mathsf{X} \rightrightarrows \mathsf{Y}$.
- X is the *input space*, Y is the *output space* of F.
- Notation: $F[x] := \{y \in \mathsf{Y} \,|\, (x, y) \in F\}$.

Recall

A *represented space* X is a set endowed with a representation $\delta_{\mathsf{X}}\colon \mathbb{N}^{\mathbb{N}} \dashrightarrow \mathsf{X}$.

Remark

We will assume every multifunction to be *total*, i.e. $F[x] \neq \emptyset$ for all $x \in \mathsf{X}$.

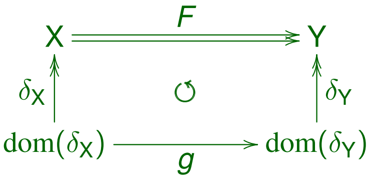
Recap

Let $F\colon X \rightrightarrows Y$ be a total multifunction.

- F is called *computable*, if there is a computable *realizer* $g\colon \mathbb{N}^{\mathbb{N}} \dashrightarrow \mathbb{N}^{\mathbb{N}}$ satisfying

$$\delta_Y g(p) \in F[\delta_X(p)] \qquad \text{for all } p \in \mathrm{dom}(\delta_X).$$

- Diagram:



- F is called *continuously realizable*, if F has a *continuous* realizer g.

Characterisation Theorem

Let X be a computable metric space and Y be an effectively Hausdorff QCB-space. Let $F\colon \mathsf{X} \rightrightarrows \mathsf{Y}$ be a total multifunction. TFAE:

(a) F is computable.

(b) There is a computable function $h\colon \mathsf{X} \to \mathcal{K}_{+\mathrm{b}}(\mathsf{Y})$ such that

$$\emptyset \neq h(x) \subseteq F[x] \quad \text{for all } x \in \mathsf{X}.$$

(c) There are computable functions $h_+ : \mathsf{X} \to \mathcal{K}_+(\mathsf{Y})$ and $h_\mathrm{b} : \mathsf{X} \to \mathcal{K}_-(\mathsf{Y})$ such that

$$\emptyset \neq h_+(x) \subseteq F[x] \cap h_\mathrm{b}(x) \quad \text{for all } x \in \mathsf{X}.$$

Remark

(a) $\Longrightarrow$ (b) holds for any represented QCB_2-space Y.

(b) $\Longrightarrow$ (a) holds for any represented space X.

(a) $\Longrightarrow$ (b) does not hold for non-metrisable spaces X.

Characterisation Theorem
Let X be a separable metric space, and Y be a QCB_2-space.
Let $F\colon X \rightrightarrows Y$ be a total multifunction.
TFAE:

(a) F has a continuous realizer.

(b)

(c) There are a lower semi-continuous function $h_+\colon X \to \mathcal{K}_+(Y)$ and an upper semi-continuous function $h_\flat\colon X \to \mathcal{K}_-(Y)$ s.t.

$$\emptyset \neq h_+(x) \subseteq F[x] \cap h_\flat(x) \quad \text{for all } x \in X.$$

If additionally $\kappa_{+\flat}$ is admissible, then (b) $\Longleftrightarrow$ (a) $\Longleftrightarrow$ (c):

(b) There is a continuous function $h\colon X \to \mathcal{K}_{+\flat}(Y)$ s.t.

$$\emptyset \neq h(x) \subseteq F[x] \quad \text{for all } x \in X.$$

Proposition

Let Y be a QCB_2-space.

- κ_{+b} is not admissible, if $\mathsf{Y} \in \omega\mathsf{Top}_2 \setminus \omega\mathsf{Top}_3$.
- κ_{+b} is admissible, if Y is quasi-normal.

Remark

- Quasi-normal space $=$ a QCB-space that arises as the sequentialisation of a normal space.
- Examples:
  - All separable metric spaces
  - The Kleene-Kreisel spaces $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}, \mathbb{N}^{\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}}, \ldots$
  - Many Hausdorff spaces used in Computable Functional Analysis
- Quasi-normal spaces have excellent closure properties:
  - cartesian closed
  - countable limits
  - countable colimits

Summary

- Semi-decidability of inequality does not imply topological Hausdorffness.
- The new notion of effective Hausdorffness implies
  - the previous notion,
  - topological Hausdorffness.
- It admits effective versions of some classical theorems from topology.
- The powerspace $\mathcal{K}_{+\mathrm{b}}(\mathsf{Y})$ allows us to characterise computable multifunctions from computable metric spaces to effective Hausdorff QCB-spaces.
- Open problem:

  Find a characterisation of computable multifunctions on input spaces that are not computable metric spaces.

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