Kolmogorov complexity and Capacitability of Dimension

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Abstract

The Hausdorff Dimension of a set of real numbers A is a numerical indication of the geometric fullness of A. Sets of positive measure have dimension 1, but there are null sets of every possible dimension between 0 and 1.

Effective Hausdorff Dimension is a variant which incorporates computability-theoretic considerations. By work of Jack and Neil Lutz, Elvira Mayordomo, and others, there is a direct connection between the the effective Hausdorff dimensions of the elements of a set A and the Hausdorff dimension of A itself. We will describe how this point-to-set principle works and how it allows for novel approaches to classical problems in Geometric Measure Theory.

Lebesgue Measure

For convenience we will work in Cantor space $\mathcal{C}$, wherein the points are infinite binary sequences $x \in 2^\omega$ and a basic open set $B(\sigma)$ consists of all extensions of a particular finite binary sequence $\sigma \in 2^{<\omega}$.

We obtain Lebesgue measure λ on $\mathcal{C}$ by setting $\lambda(B(\sigma)) = 1/2^{|\sigma|}$, where $|\sigma|$ denotes the length of σ, and to the σ-algebra of measurable sets. Then, when A is measurable,

$$\lambda(A) = \inf\left\{ \sum_{k=1}^{\infty} \lambda(B(\sigma_k)) : \begin{array}{l} (\sigma_k)_{k\in\mathbb{N}} \text{ is a sequence from } 2^{<\omega} \\ \text{with } A \subseteq \bigcup_{k=1}^{\infty} B(\sigma_k) \end{array} \right\}.$$

Regularity of Lebesgue Measure

Remark

If A is measurable, then

$$\begin{aligned}\lambda(A) &= \inf\{\lambda(O) : O \textit{ is open and } A \subseteq O\} \\ &= \sup\{\lambda(C) : C \textit{ is closed and } C \subseteq A\}\end{aligned}$$

In other words, the measure of A is carried by the measures of its closed subsets.

Randomness

formulated by measure

Definition

A sequence x is *Martin-Löf random* iff it does not belong to any effectively-null G_δ set. Precisely, if $(O_k : k \in \mathbb{N})$ is a uniformly computably enumerable sequence of open sets such that for all k, O_k has measure less than $1/2^k$, then $x \notin \cap_{k \in \mathbb{N}} O_k$.

Randomness

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Definition

- For $\sigma \in 2^{<\omega}$, $K(\sigma)$ is the length of the shortest program which outputs σ and then halts, in a universal prefix-free listing of programs.
- A sequence $x \in 2^\omega$ is *algorithmically incompressible* iff there is a C such that for all ℓ, $K(x \upharpoonright \ell) > \ell - C$, where K denotes prefix-free Kolmogorov complexity.

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Theorem (Schnorr 1973)

x is Martin-Löf random iff it is algorithmically incompressible.

Random Sequences and Closed Sets

A closed set C in 2^ω can be represented as the set of infinite paths in a subtree T of $2^{<\omega}$. (The terminal nodes of T index the basic open sets that constitute the complement of C.)

When T is computable, then C is a Π^0_1 class. Otherwise, C is Π^0_1 relative to T.

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Theorem (Folklore, I first heard it from Yu Liang)

- *If C is Π^0_1 relative to T, then C has positive measure iff C has an element which is Martin-Löf random relative to T.*
- *An arbitrary set A has positive measure iff for all T there is an element of A which is Martin-Löf random relative to T.*

Point-to-Set for Measure

An expository device for what is to follow

For every real, A has an element which is Martin-Löf random relative to that real iff A has positive measure.

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For every real, A has an element which is Martin-Löf random relative to that real iff A has positive measure.

- One proof is to use the regularity of measure and invoke the result on the previous slide.
- Alternatively, one can return to the definition of measure: If A has measure zero, then for every $\epsilon > 0$, there is an open cover of A of measure less than ϵ. Take a real uniformly coding a sequence of covers measures less than $1/2^n$. No element of A is Martin-Löf random relative to that real. The other direction is equally clear.

Hausdorff Dimension

Define a family of outer measures, parameterized by $d \in [0,1]$. For $A \subseteq 2^\omega$,

$$\mathcal{H}^d(A) = \lim_{r \to 0} \inf \left\{ \sum_i \frac{1}{2^{|\sigma_i|\, d}} : \begin{array}{l} \text{there is a cover of } A \text{ by balls} \\ B(\sigma_i) \text{ with } 1/2^{|\sigma_i|} < r \end{array} \right\}.$$

Definition

The *Hausdorff dimension* of A is as follows.

$$\begin{aligned} \dim_{\mathrm{H}}(A) &= \inf\{d \geq 0 : \mathcal{H}^d(A) = 0\} \\ &= \sup\left(\{d \geq 0 : \mathcal{H}^d(A) = \infty\} \cup \{0\}\right) \end{aligned}$$

Frostman's Lemma

Theorem (Frostman 1935, Besicovitch and Davies 1952 (independently))

For A an analytic subset of 2^ω,

$$\dim_H(A) = \sup\left\{ s : \begin{array}{l} \text{there is a Borel measure } \mu \text{ such that } \mu(A) > 0 \\ \text{and for all } \sigma \in 2^{<\omega},\ \mu(B(\sigma)) \leq \left(\frac{1}{2^{|\sigma|}}\right)^s \end{array} \right\}$$

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Corollary

If A is an analytic subset of 2^ω and $\dim_H(A) = d$, then for every $s < d$ there is a closed set $C_s \subseteq A$ such that $s \leq \dim_H(C) \leq d$.

In other words, the Hausdorff dimension of analytic A is carried by the dimensions of its closed subsets.

Effective Hausdorff Dimension

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Definition

- For $A \subseteq 2^\omega$, define A has *effective s-dimension Hausdorff measure* 0 iff there is a uniformly computably enumerable sequence of open sets $O_i = \bigcup_j B(\sigma_{i,j})$ such that for each i, $A \subseteq O_i$ and $\sum_j (1/2^{|\sigma_{i,j}|})^s < 1/2^i$.
- The *effective Hausdorff dimension* $\dim_{\mathrm{H}}^{\mathit{eff}}(A)$ of A is the infimum of those s such that A has effective s-dimension Hausdorff measure 0.

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Remark

- *For all A,* $\dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{H}}^{\mathit{eff}}(A)$
- *If x is Martin-Löf random then* $\dim_{\mathrm{H}}^{\mathit{eff}}(\{x\}) = 1$.

Effective Hausdorff Dimension

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Definition

A sequence $x \in 2^\omega$ is *algorithmically compressible by a factor of s* iff there is a C such that there are infinitely many ℓ such that $K(x \upharpoonright \ell) \leq s\,\ell - C$, where K denotes prefix-free Kolmogorov complexity.

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Theorem (Mayordomo 2002)

For any $x \in 2^\omega$, $\dim_{\mathrm{H}}^{\mathit{eff}}(\{x\})$ is the infimum of the s such that x is algorithmically compressible by a factor of s.

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- We will abbreviate and write $\dim_{\mathrm{H}}^{\mathit{eff}}(x)$ for $\dim_{\mathrm{H}}^{\mathit{eff}}(\{x\})$.
- We can relativize to a real z and write $\dim_{\mathrm{H}}^{\mathit{eff}(z)}(x)$.

Frostman's Lemma Revisited

Theorem (Reimann 2008)

Suppose that $\dim_{\mathrm{H}}^{\mathit{eff}}(x) = d$*. For all* $s < d$*, there is an* s*-regular Borel measure* μ *such that* x *is Martin-Löf random for the measure* μ*.*

Point-to-Set for Hausdorff Dimension

Theorem (J. Lutz and N. Lutz 2017)

For $A \subseteq 2^\omega$, the Hausdorff dimension of a set A is equal to
the infimum over all $B \subseteq \mathbb{N}$
of the supremum over all $x \in A$
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- Unlike in the discussion of measure, since we are not assuming that A is analytic, we cannot immediately move to a closed subset of A.
- We can return to the definition of Hausdorff dimension in terms of open covers. One direction of the above is obtained by considering the reals that can compute appropriate open covers of A. The other direction can be proven by using the relativized version of the fact that the set of reals with effective Hausdorff dimension s has Hausdorff dimension s.

Co-analytic Sets

Consistency results

We will look at these phenomena in Gödel's universe of constructible sets L, consisting of those sets obtained from the empty set and transfinitely iterating first order definability.

In what follows, assume that every set is constructible, i.e. $V = L$.

Co-analytic Sets

Working in $V = L$

Definition

Define P by

$$P = \left\{ x : \begin{array}{l} x \text{ can compute a representation of the ordinal at} \\ \text{which } x \text{ is constructed} \end{array} \right\}$$

Theorem (Original reference unknown to me)

- *P is co-analytic.*
- *P is not countable.*
- *P has no perfect subset.*

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$\dim_H(P) = 1$.

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Consequently, the following are consistent with *ZFC*

- The Hausdorff dimensions of co-analytic sets are not carried by their closed subsets.
- The Frostman/Besicovitch-Davies Theorem does not extend from analytic to co-analytic sets.

Applying Point-to-Set Reasoning

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Step 1. There is an infinite computable set $S \subseteq \mathbb{N}$ such that for all z and for all x, if x is Martin-Löf random relative to z and y is equal to x at all places not in S then $\dim_{\mathrm{H}}^{eff(z)}(y) = 1$.

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In fact, S could be the iterated powers of 2. To verify the claim, use Mayordomo's theorem and estimate the compressibility of y relative to z.

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In fact, S could be the iterated powers of 2. To verify the claim, use Mayordomo's theorem and estimate the compressibility of y relative to z.

Step 2. By the Lutz and Lutz theorem, it is sufficient to show that for every z there is a y in P such that $\dim_{\mathrm{H}}^{eff(z)}(y) = 1$.

Applying Point-to-Set Reasoning

Working in $V = L$

Step 3. Suppose that $z \in 2^\omega$ is given.

- Let x be Martin-Löf random relative to z.
- Let $m \in P$ be such that m can compute x and z.
- Let y be the result of replacing the bit values of x on the elements of S by the bit values of m.

Then, m can compute the ordinal at which y is constructed and y can compute m. Thus, $y \in P$.

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Step 4. Conclude, $\dim_H(P) = 1$, as required.

Comments

- J. Lutz, N. Lutz and Don Stull have other applications of effective Hausdorff dimension within Geometric Measure Theory.
- This mode of argument is in an early phase. It would be interesting to see whether/how it develops.

The End