PA relative to an enumeration oracle

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Enumeration reducibility

Definition (Friedberg and Rogers 1959)

$A \leqslant_e B$ if there is c.e. set W such that

$$A = W(B) = \{x \mid \exists v(\langle x, v\rangle \in W \;\&\; D_v \subseteq B\}$$

Proposition. A is c.e. in B if and only if $A \leqslant_e B \oplus B^c$.

Unlike the relation "c.e. in", the relation $\leqslant_e$ is transitive. It gives rise to the structure of the enumeration degrees $\mathcal{D}_e$.

The Turing degrees properly embed into $\mathcal{D}_e$ as the *total degrees*, degrees of sets of the form $A \oplus A^c$.

Relative to an enumeration oracle

When we relativize a class of objects with respect to a Turing oracle A, we usually replace "c.e." by "c.e. in A".

Example

For $W \subseteq 2^{<\omega}$ let $[W] = \{X \in 2^\omega \mid \exists \sigma \in W(\sigma \preceq X)\}$.

P is a Π^0_1 class is there is a c.e. set $W \subseteq 2^{<\omega}$ such that $P = 2^\omega \setminus [W]$.

P is a $\Pi^0_1(A)$ class is there is a c.e. in A set $W \subseteq 2^{<\omega}$ such that $P = 2^\omega \setminus [W]$.

We can extend this relation to enumeration oracles by replacing "c.e. in A" by "$\leqslant_e A$".

Definition

P is a $\Pi^0_1\langle A\rangle$ class if there is some $W \leqslant_e A$ such that $P = 2^\omega \setminus [W]$.

Note that a $\Pi^0_1\langle A \oplus A^c\rangle$ class is just a $\Pi^0_1(A)$-class.

The relation “PA above”

Recall that for Turing oracles A and B we say that B *is PA above* A if B computes a member of every nonempty $\Pi^0_1(A)$ class.

Definition

$\langle B \rangle$ *is PA relative to* $\langle A \rangle$ if B enumerates a member of every nonempty $\Pi^0_1\langle A \rangle$ class.

We treat the elements of a $\Pi^0_1\langle A \rangle$ class P as total objects! B enumerates a member of P, if there is some $X \in P$ such that $X \oplus X^c \leqslant_e B$.
If P is a $\Pi^0_1\langle A \rangle$ class then so are $\{X^c \mid X \in P\}$ and $\{X \oplus X^c \mid X \in P\}$.

Thus, B is PA above A if and only if $\langle B \oplus B^c \rangle$ is PA above $\langle A \oplus A^c \rangle$.

Good oracles: the continuous degrees

The *continuous degrees* were introduced by Miller (2004) to capture the algorithmic content of points in computable Polish spaces. They form a proper (definable) subclass of the enumeration degrees and properly extend the total degrees.

Theorem (Miller 2004).

1. If $\mathbf{a}$ is a nontotal continuous degree then the set total degrees bounded $\mathbf{a}$ is a *Scott set*, i.e. a Turing ideal closed under the relation PA above.
2. For total degrees $\mathbf{y}$ is PA above $\mathbf{x}$ if an only if there is some non-total continuous degree $\mathbf{a}$ with $\mathbf{x} < \mathbf{a} < \mathbf{y}$.

Good oracles: the continuous degrees

Theorem (Andrews, Igusa, Miller, S 2019). A has continuous degree if and only if A is *codable*—there is a nonempty $\Pi^0_1\langle A\rangle$ class C_A such that every member of C_A uniformly enumerates A.

Corollary.

1. If A has continuous degree then $\langle A\rangle$ is not PA relative to $\langle A\rangle$—not *$\langle$self$\rangle$-PA*.
2. If A has continuous degree and $\langle B\rangle$ is PA relative to $\langle A\rangle$ then $A \leqslant_e B$—A is *PA bounded*.
3. There is a *universal $\Pi^0_1\langle A\rangle$-class P*: a nonempty class whose every member is PA relative to $\langle A\rangle$.

Question. Are there any bad oracles?

Bad oracles: $\langle$self$\rangle$-PA oracles

Theorem (Miller, Soskova 2014). There are $\langle$self$\rangle$-PA degrees.

Proposition. If A is $\langle$self$\rangle$-PA then A cannot have a universal class.

Proof: If A is $\langle$self$\rangle$-PA and P is universal then A enumerates some $X \in P$. But now every $\Pi^0_1(X)$-class is a $\Pi^0_1\langle A\rangle$ class and X computes a member of it. □

Question.

1. Can $\langle$self$\rangle$-PA degrees be PA bounded?
2. Can non-continuous degrees have a universal class?

Continuous = PA bounded

Theorem(Franklin, Lempp, Miller, Schweber, and S 2019). The continuous degrees are exactly the PA bounded enumeration degrees.

Proof idea: If A does not have continuous degree, we use the fact that A is not codable to produce a nested sequence of $\Pi^0_1\langle A\rangle$-classes $\{P_e\}_{e<\omega}$ such that every member of P_e computes a member of each nonempty $\Pi^0_1\langle A\rangle$ indexed by a number less than e but does not enumerate A via Γ_e. We then take $X \in \bigcap P_e$.

Question.

1. Can $\langle$self$\rangle$-PA degree be PA bounded? No!
2. *Can non-continuous degrees have a universal class?*

Other ways to have a universal class

Definition

An enumeration oracle $\langle A\rangle$ is *low for PA* if every set $X \oplus X^c$ that is PA (in the Turing sense) is PA relative to $\langle A\rangle$.

Total non-computable oracles cannot be low for PA: they are PA bounded, but there is a minimal pair of PA degrees.

In fact, low for PA oracles are *quasiminimal* (hence disjoint from continuous degrees).

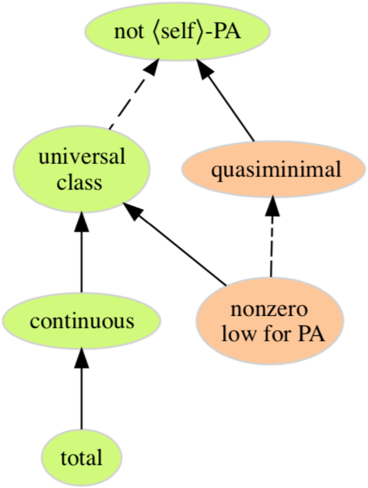
Low for PA oracles have a universal class (e.g. DNC_2).

Theorem(Goh, Kalimullin, Miller, S). $\langle A\rangle$ is low for PA if and only if every nonempty $\Pi^0_1\langle A\rangle$ class has a nonempty Π^0_1 subclass.

Theorem(GKMS). The following classes of e-oracles are low for PA.

1. The 1-generic degrees.
2. Halves of nontrivial $\mathcal{K}$-pairs.

The picture so far



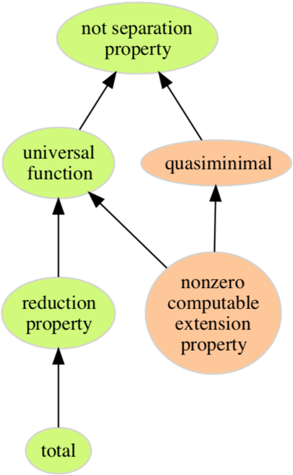
Notions from descriptive set theory

Kalimullin and Puzarenko in 2005 defined and studied the following classes of enumeration oracles with definitions inspired from descriptive set theory and classical computability theory:

1. Oracles with the *reduction property*;
2. Oracles with the *uniformization property*;
3. Oracles with the *separation property*;
4. Oracles with the *computable extension property*;
5. Oracles with a *universal function*.

They showed:

Kalimullin and Puzarenko's theorem



The reduction property

X has the *reduction property* if whenever $A, B \leqslant_e X$ there are disjoint $A_0, B_0 \leqslant_e X$ with $A_0 \subseteq A$, $B_0 \subseteq B$, and $A_0 \cup B_0 = A \cup B$;

Example

Kleene's O has the reduction property because $A \leqslant_e O$ if and only if A is Π^1_1.

What goes wrong if we try to build a universal $\Pi^0_1\langle X\rangle$ class?

We want to construct a $\Pi^0_1\langle X\rangle$ class U such that if $P_e\langle X\rangle \neq \varnothing$ then the e-th column in any member of U codes a member of $P_e\langle X\rangle$.

We would like to define U as the class of separators for

1. The set A of all $\langle e, \sigma\rangle$ such that all extensions of $\sigma 0$ leave $P_e\langle X\rangle$ *first*.
2. The set B of all $\langle e, \sigma\rangle$ such that all extensions of $\sigma 1$ leave $P_e\langle X\rangle$ *first*.

If X is not total then we don't have a notion of *first*!

But then for σ with no extension in P_e we will have $\langle e, \sigma\rangle \in A \cap B$.

The reduction property lets us solve exactly this problem!

Theorem(GKMS). The reduction property implies having a universal class.

The separation property

X has the *separation property* if for every pair of disjoint sets $A, B \leqslant_e X$ there is a separator C such that $A \subseteq C$, $B \subseteq C^c$, and $C \oplus C^c \leqslant_e X$.

Note that the set of all separators C for sets $A, B \leqslant_e X$ is a $\Pi^0_1\langle X\rangle$ class.

Definition

A $\Pi^0_1\langle X\rangle$ class P is a *separation class* if $P = \{C \mid A \subseteq C \ \&\ B \subseteq C^c\}$ for some disjoint $A, B \leqslant_e X$. Call such classes *Sep*$\langle X\rangle$ for short.

Proposition. X has the separation property if and only if X enumerates a path in every Sep$\langle X\rangle$ class.

If X is $\langle$self$\rangle$-PA then X has the separation property.

Computable extension property

X has the *computable extension* property if every partial function φ with $G_\varphi \leqslant_e X$ has a (partial) computable extension $\psi \subseteq \varphi$.

Theorem (GKMS). The following are equivalent:

1. X has the computable extension property.
2. Every $\{0,1\}$-valued function with graph reducible to X has a computable $\{0,1\}$-valued extension.
3. If $A \leqslant_e X$ and $B \leqslant_e X$ are disjoint then there are disjoint c.e sets C and D such that $A \subseteq C$ and $B \subseteq D$.
4. Every set Y with PA degree computes a member of every $\mathrm{Sep}\langle X \rangle$ class.

And so if X is low for PA then X has the computable extension property.

A mystery solved by introducing uniformity

X has a *universal function* if there is a partial function U with $G_U \leqslant_e X$ such that if φ is a partial function with $G_\varphi \leqslant_e X$ then for some e we have that $\varphi = \lambda x.U(e, x)$

Question. This should be an analog of having a universal class, but how?

We defined a *universal* $\Pi^0_1\langle X\rangle$*-class* to be a nonempty class whose every member is PA relative to $\langle X\rangle$, i.e. enumerates a path in every nonempty $\Pi^0_1\langle X\rangle$ class.We will adjust this definition introducing a little uniformity:

Definition

P is a *universal* $\Pi^0_1\langle X\rangle$*-class* if for every nonempty $\Pi^0_1\langle X\rangle$ class Q there is a uniform procedure that produces a path from Q relative to every member of P.

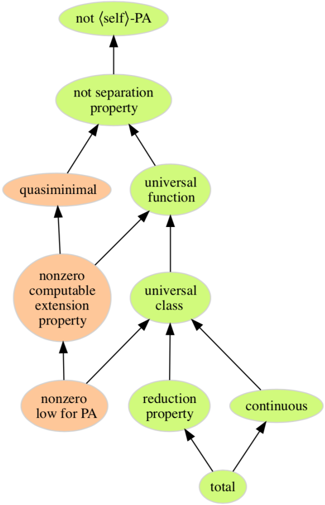
In all cases we looked at so far, that is the case: total degrees, the continuous degrees, the low for PA degrees, the oracles with the reduction property!

Universal for Sep$\langle X \rangle$ classes

Theorem (GKMS). The following are equivalent

1. X has a universal function;
2. X has a $\{0,1\}$-valued universal function U for $\{0,1\}$-valued partial functions φ with $G_\varphi \leqslant_e X$;
3. There is a $\Pi^0_1\langle X \rangle$ class P such that for every Sep$\langle X \rangle$-class Q there is a uniform procedure that produces a path from Q relative to every member of P. (This class can be chosen as a separating class.)

A summary of the results by Goh, Kalimullin, Miller, and Soskova



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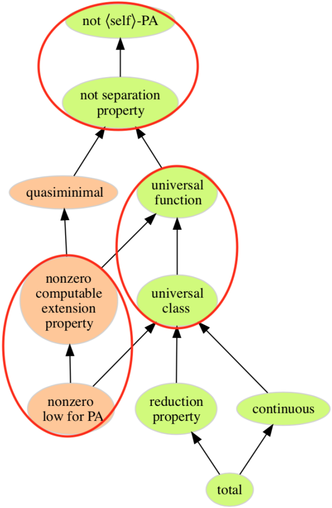


All of the arrows are strict! A forcing notion

Let $f(n) = 2^n$. We identify ω with $f^{<\omega}$—the set of sequences $\sigma \in \omega^{<\omega}$ such that $\sigma(n) < 2^n$ for all $n < |\sigma|$.

A forcing condition is a pair $\langle T, \varepsilon \rangle$:

- T is a finite subtree of $f^{<\omega}$ of height $|T|$;
- $\varepsilon \in (0, 1)$ is rational.

$\langle S, \delta \rangle \leqslant \langle T, \varepsilon \rangle$ if and only if

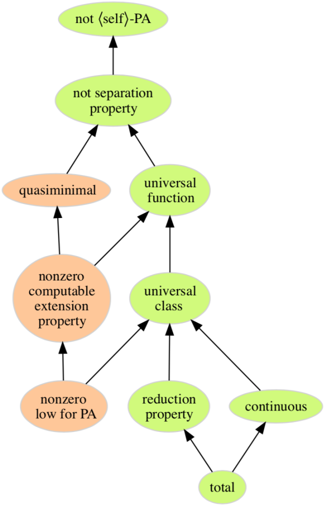
- $T = S \upharpoonright |T|$,
- $\delta \leqslant \varepsilon$, and
- for every $\sigma \in S$ with $|T| \leqslant |\sigma| < |S|$, at least $\lceil (1 - \varepsilon) \cdot 2^{|\sigma|} \rceil$ of its immediate successors lie in S.

If $\mathcal{F}$ is a filter in this partial order then let $G = \bigcup_{\langle T, \varepsilon \rangle \in \mathcal{F}} T$ and $A_G = f^{<\omega} \smallsetminus G$.

Lemma. If G is sufficiently generic, then A_G has the computable extension property.

Lemma. If G is sufficiently generic, then A_G does not have a universal class.

Thank You!



Open questions.

1. Is the extra uniformity that we added to the definition of universal class necessary?
2. If A has a universal class does A have a separating class that is universal?
3. Is the relation PA relative to an enumeration oracle definable?
4. X has the *effective inseparability property* if there are disjoint sets $A, B \leqslant_e X$ and a function ψ with $G_\psi \leqslant_e X$ such that if $A \subseteq W_x(X)$ and $B \subseteq W_y(X)$ are disjoint then $\psi(x, y) \downarrow\notin W_x(X) \cup W_y(X)$. How does this class fit in with the rest?

Visit http://zoo.ludovicpatey.com/ to build your own pretty diagram!