On definability of c.e. degrees in the 2-c.e. degree structures

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Oberwolfach, April 27, 2021

Definitions and notations

All sets are subsets of $\omega = \{0, 1, 2...\}$. Thus, let $A, B \subset \omega$.

- $A \leq_T B$ if there is an algorithm that allows to answer the questions "$x \in A$?", using B as an oracle.
- $A \leq_m B$ if there is a computable function f such that $x \in A \Longleftrightarrow f(x) \in B$.
- Clearly, $A \leq_m B \implies A \leq_T B$.

Definitions and notations

- If $A \leq_r B$ and $B \leq_r A$ then $A \equiv_r B$.
- Let $\deg_r(A) = \{B \mid A \equiv_r B\}$.
- Here, $r \in \{m, T\}$.

The Coopers theorem

Theorem (Cooper, 1971)

There is a 2-c.e. set with proper 2-c.e. Turing degree.

Remark. A 2-c.e. degree is proper if it doesn't contain a c.e. set. In particular, the constructed set has proper 2-c.e. m-degree.

As a corollary, the universes for c.e. and 2-c.e. degree structures are different. And clearly, c.e. degrees form a substructure in the corresponding 2-c.e. degrees.

Motivations and goals

- To investigate the 2-c.e. degree structures.
- To investigate model-theoretic properties of c.e. and 2-c.e. degrees (in different settings).
- To study relationship between c.e. and properly 2-c.e. degrees (in different settings).

Motivations and goals

Open question (Cooper, 2002; Arslanov, 2009)

Is the class of c.e. Turing degrees definable in the partial ordering of 2-c.e. Turing degrees?

Related questions:

- The same questions for m-degrees.
- A weaker version of the question involving parameters.
- A weaker version of the question involving additional predicates.
- The case of low c.e. and 2-c.e. degrees.

Definability

Let $\mathcal{A}$ be a structure, and B be a subset of $|A|$.

Definition

The class B is definable in $\mathcal{A}$ if there exists a formula $\varphi(x)$ of the first order language such that for all $a \in |A|$ it holds

$$\mathcal{A} \models \varphi(a) \Leftrightarrow a \in B$$

- As $\mathcal{A}$ we consider $(\mathbf{D}, \leq)$, where $\mathbf{D}$ is the corresponding 2-c.e. degrees, and $\leq$ is induced by the same reducibility.
- As B we consider $\mathbf{R}$, the c.e. degrees.
- In φ, there can be additional fixed variables $c_1, c_2, \dots \in |A|$ called parameters.

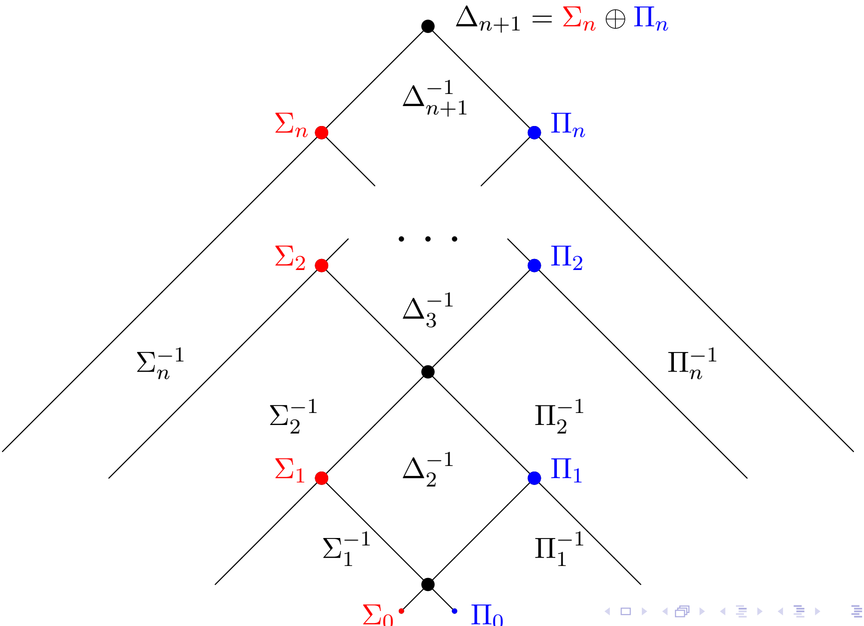
Section 1

Definability for m-degrees

A brief history

- m-degrees were actively studied since 1970 (by Degtev, Denisov, Ershov, Nies, Lachlan, Selivanov, etc.)
- The most attention was received by c.e. m-degrees and by all m-degrees.
- In general the structures of m-degrees found out to have many good properties, in particular, much better then the structures of T-degrees.
- For example, Σ_n^{-1} m-degrees have the greatest (universal) element (by Ershov), Δ_n^{-1} m-degrees have the greatest element (by Selivanov), for any fixed $n > 0$.

Picture



$\Delta_{n+1} = \Sigma_n \oplus \Pi_n$
Δ_{n+1}^{-1}
Σ_n
Π_n
$\cdots$
Σ_2
Π_2
Δ_3^{-1}
Σ_n^{-1}
Π_n^{-1}
Σ_2^{-1}
Π_2^{-1}
Σ_1
Δ_2^{-1}
Π_1
Σ_1^{-1}
Π_1^{-1}
Σ_0
Π_0

Facts and folklore

- c.e. m-degrees form an ideal in 2-c.e. m-degrees
- c.e. and co-c.e. m-degrees are isomorphic.
- c.e. and 2-c.e. m-degrees form a distributive upper semilattice (by Ershov, Lachlan, Selivanov). The same holds for c.e. wtt-degrees, but doesn't hold for 2-c.e. wtt-degrees, and for c.e. and 2-c.e. Turing degrees.
- The greatest c.e. m-degree is not splittable (by Lachlan), thus Δ- and Σ-(Π-) levels are not elementarily equivalent. The result has a direct generalization to 2-c.e. m-degrees.

Facts and folklore

- Given 2-c.e. set $A = A_0 - A_1$, let $A_0 = rng(f)$ for some computable 1-1 f, then $L(A) = f^{-1}(A_1)$ is Lachlan's set for A.
- $L(A)$ is c.e.
- $\overline{L(A)} \leq_m A$
- If $L(A)$ is c.e. then A is 2-c.e., and if $L(A)$ is computable then A is c.e.
- Then below any proper 2-c.e. m-degree there exists a noncomputable co-c.e. m-degree.

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Elementary difference

Theorem (Ershov and Lavrov, 1973)

Given noncomplete c.e. set B there exists a c.e. set $A \not\leq_m B$ which is minimal

Corollary

c.e. and 2-c.e. m-degrees are not elementarily equivalent

Note that for we can take U_{Δ_2} as B, then any set $A \not\leq_m B$ would be proper 2-c.e. and have a noncomputable element below it.

Remark. The theorem was proved for much more general case and for the c.e. setting. For details see [Erhov Yu.L., Lavrov I.A. Upper semilattice L(S), Algebra i Logica, 1973, Vol.12, No.2, P.167-189]

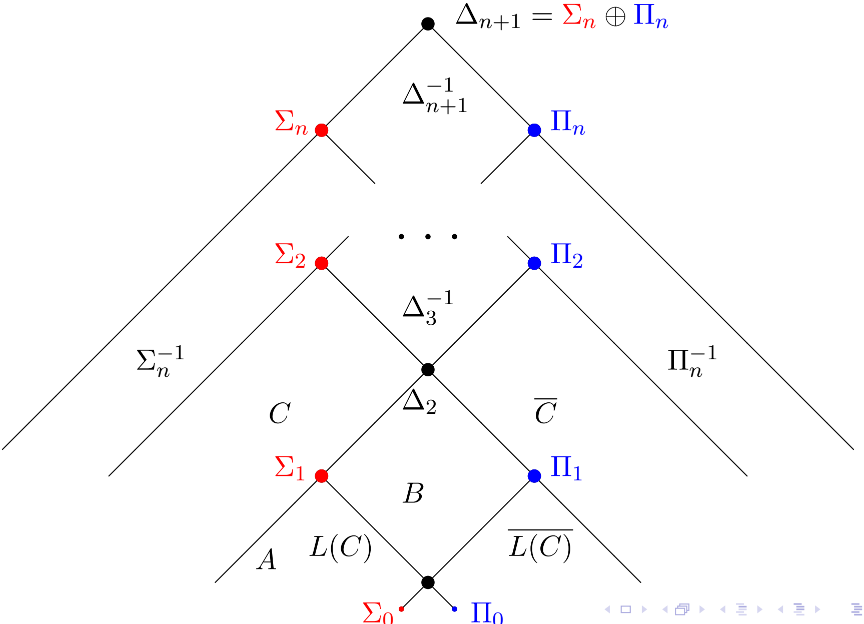
The main structural theorems

Theorem 1 (Ng, Yamaleev)

Given $k, n > 0$, given any Σ_k^{-1} set B such that $U_{\Sigma_n} \nleq_m B$, there exists a Σ_n^{-1} set $A \nleq_m B$ such that for any $W <_m A$ it holds that $W \leq_m U_{\Delta_n}$.

- noncomplete $B \leftrightsquigarrow U_{\Sigma_n} \nleq_m B$
- minimal $A \leftrightsquigarrow A$ is minimal cover for U_{Δ_n}
- c.e. $A \nleq_m B \leftrightsquigarrow \Sigma_n^{-1}$ set $A \nleq_m B$

Picture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C
Δ_2
$\overline{C}$
Σ_1
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B
A
$L(C)$
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Σ_0
Π_0

The main structural theorems

Corollary (Ng, Yamaleev)

Given $k, n > 0$, given any Σ_k^{-1} set B such that $U_{\Delta_{n+1}} \nleq_m B$, there exists a Δ_{n+1}^{-1} set $A \nleq_m B$ such that for any $W <_m A$ it holds that $W \leq_m U_{\Delta_n}$.

Theorem 2 (Ng, Yamaleev)

Given $n > 0$, there exists a set A of properly Σ_{n+1}^{-1} degree such that for any $W \in \Sigma_n^{-1}$ if $W \leq_m A$ then $W \leq_m U_{\Delta_n}$.

The main structural theorems, 2-c.e. setting

Corollary (Ershov, Lavrov, 1973)

Given noncomplete (in Δ_2^{-1} m-degrees) set B there exists a 2-c.e. set $A \not\leq_m B$ such that A has a minimal m-degree (moreover, it will be either c.e. or co-c.e.)

Corollary (from Theorem 2 (Ng, Yamaleev))

There exists a set A of properly 2-c.e. m-degree such that for any c.e. W if $W \leq_m A$ then W is computable (i.e., A form minimal pair with the greatest c.e. degree).

Intuitive description

- The first part says we can build minimal m-degrees avoiding arbitrary (noncomplete) lower cones.
- The second part says that for all c.e. m-degrees we can find a half minimal pair in the 2-c.e. m-degrees.
- Note also that we cannot do it for co-c.e. m-degrees using a unique 2-c.e. degree.

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- Note also that we cannot do it for co-c.e. m-degrees using a unique 2-c.e. degree.

The corollaries

- The degree structures of c.e. and 2-c.e. m-degrees are not elementarily equivalent (and it works for all higher levels).
- The m-degree of universal Δ_2^{-1}-set is definable in 2-c.e. m-degrees.
- The complementary Theorem 2 allows to distinguish the greatest c.e. from the greatest co-c.e. m-degree.

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Definability of c.e. in 2-c.e.

- $\theta(x) := \forall b \; [x \not\leq b \Rightarrow \exists a \; (a \not\leq b \wedge \forall w \; [w < a \Rightarrow w \leq 0])]$
- $\psi(x) := \theta(x) \wedge \forall z[x < z \Rightarrow \neg\theta(z)]$,
- Thus, $\psi(x)$ is true in Σ_2^{-1} iff $x = U_{\Delta_2}$

- $\varphi(x, y) := \exists u \; \psi(u) \wedge x \cup y = u \wedge [\forall x_1 \; \forall y_1 (x_1 < x \Rightarrow x_1 \cup y < u) \wedge (y_1 < y \Rightarrow x \cup y_1 < u)]$
- Thus, $\varphi(x, y)$ defines the pair of U_{Σ_1} and U_{Π_1} but cannot distinguish them.

- $\varphi^{\Sigma}(x) := \exists y \; (\varphi(x, y) \wedge \exists z \; \forall w[z \not\leq x \cup y \wedge w < z \wedge w \leq x \Rightarrow w \leq 0])$

Complexity of the formulas

- Elementarily difference of c.e. and 2-c.e.: Σ^0_2
- Definability of c.e. in 2-c.e.: Σ^0_4
- For higher levels the complexity grows incredibly. For instance, in Σ^{-1}_n we define in the following ordering: U_{Δ_2}, U_{Δ_3}, ..., U_{Δ_n}, then $U_{\Sigma_{n-1}}$, $U_{\Sigma_{n-2}}$, ..., U_{Σ_1}.

Questions

- Is Σ_1^{-1} level definable in the structure of Σ_ω^{-1}-level?
- Could the same approach work for infinite levels? (probably with parameters)
- What to do with the limit levels?

Section 2

A weaker definability for Turing degrees

Approaches

to the problem of definability of c.e. Turing degrees in partial ordering of 2-c.e. Turing degrees.

Proposed by Arslanov and Yamaleev (2018)

1. Density of double bubbles
2. Nonspilliting pairs
3. Lachlan sets and degrees
4. Isolation from side

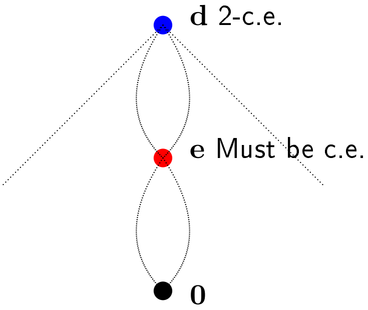
Definable singletons

Definition (Arslanov, Kalimullin, and Lempp, 2010)

Let $\mathbf{e}, \mathbf{d}$ be 2-c.e. degrees such that $\mathbf{0} < \mathbf{e} < \mathbf{d}$. We say that these degrees form a *double bubble* (also, *a bubble pair*, *2-bubble*, *bubble*) in 2-c.e. degrees if any 2-c.e. degree $\mathbf{u} < \mathbf{d}$ is comparable with $\mathbf{e}$. Also, we say that $\mathbf{d}$ is the top of bubble, and $\mathbf{e}$ is the middle of bubble. By default, we consider bubbles in 2-c.e. degrees.

- The degree $\mathbf{e}$ must be c.e.
- The degree $\mathbf{d}$ is an exact 2-c.e. degree.
- The degree $\mathbf{d}$ is not splittable avoiding upper cone of $\mathbf{e}$.

Approach 1. The picture.



Approach 1. The idea.

- To show that between any two c.e. degrees we can find a degree $\mathbf{e}$.
- Then any c.e. degree has a splitting where the both parts are middles of bubbles.
- Such splitting doesn't exists for properly 2-c.e. degrees.

Approach 1. The results.

- [Liu, Wu, Yamaleev, 2015]The exact 2-c.e. degrees are downward dense.
- [Andrews, Kuyper, Lempp, Soskova, Yamaleev, 2017] There exists a nonzero c.e. degree such that no double bubble can be found below it.
- Conjecture [Arslanov, Yamaleev, 2018] The middles of double bubbles can be found below any nonzero c.e. degree, moreover it can be combined with lower cone avoidance.

Approach 1. Conclusion.

- Definable middle of bubbles with fairly “easy” construction.
- Even if we cannot prove the density. The middles of bubbles is still a reliable class of c.e. degrees. And can be combined with downward density and cone avoidance.
- Can a middle of bubble be constructed above any low or superlow c.e. degree?

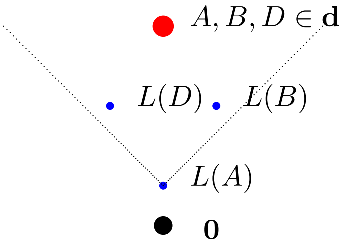
Approach 3. Idea.

- To use $L(D)$ which reflects to enumerability properties of a 2-c.e. set D. Then consider a collection of $L(B)$ such that $B \equiv_T D$.
- Make a connection between the associated degrees $L(B)$ and the degree of D.
- The good case is when for each properly 2-c.e. degree of B the collection of the degrees of $L(B)$ is bounded from below by some nonzero c.e. degree.

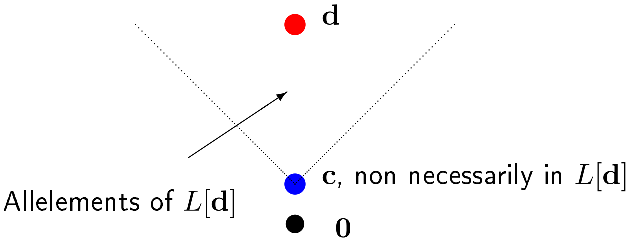
Approach 3. Results.

- Series results by Ishmukhametov [1999,2000] and by Fang, Liu, Wu, Yamaleev [2013-2019] showed that different distributions for $L(B)$ are possible.
- In particular, there is a properly 2-c.e. degree with unbounded collection of its associated degrees of $L(B)$.
- Also: if $D \equiv_T B$ and have a proper 2-c.e. degree then $L(D)$ and $L(B)$ cannot form a minimal pair.

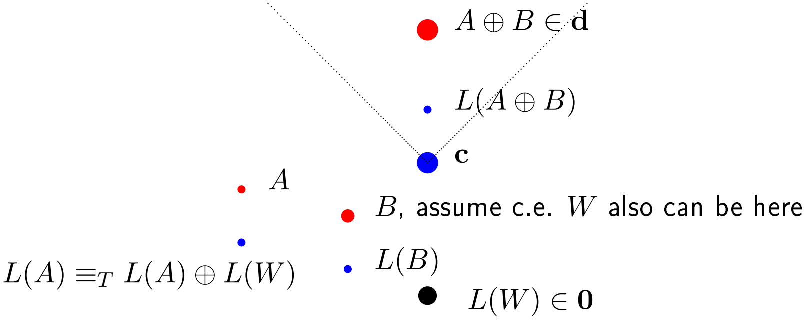
Approach 3. Picture.



Lower bounds for $L[\mathbf{d}]$?



Approach 3. Motivation



Lachlan degrees

- We say that $L[\mathbf{d}] = \{\deg(L(D))|\ D \in \mathbf{d}\}$ is a spector of Lachlan degrees for $\mathbf{d}$.

- Let $R[\mathbf{d}] = \{W|\ W \text{c.e. and } \mathbf{d} \text{ is } CEA(W)\}$.
- Then, clearly, $L[\mathbf{d}] \subset R[\mathbf{d}]$.

- The following theorem allows to obtain $R[\mathbf{d}] \subset L[\mathbf{d}]$.

Theorem (Arslanov, LaForte, Slaman, 1998)

Given ω-c.e. degree $\mathbf{d}$, which is $CEA(\mathbf{c})$ for some c.e. degree $\mathbf{c}$. Then there exists a 2-c.e. set $D \in \mathbf{d}$ such that D is $CEA(\mathbf{c})$. *Moreover, the degree* $\mathbf{c}$ *contains* $L(D)$.

Lachlan degrees

- [Ishmukhametov, 1999]. There exists a noncomputable 2-c.e. degree $\mathbf{d}$ such that $L[\mathbf{d}] = [\mathbf{c}, \mathbf{b}]$ for some noncomputable c.e. degrees $\mathbf{c}$ and $\mathbf{b}$. In particular, it can be $\mathbf{c} = \mathbf{b}$.
- [Arslanov, Kalimullin, Lempp, 2010]. There exists 2-c.e. degrees $\mathbf{c} < \mathbf{d}$ such that they form bubble. In particular, it also holds $L[\mathbf{d}] = \{\mathbf{c}\}$.
- For such bubble pairs the degree $\mathbf{c}$ is definable.

- [Ishmukhametov, 1999]. Question. Does $L[\mathbf{d}]$ alsways contain a least element for any $\mathbf{d}$?
- [Ishmukhametov, 2000]. There eixsts a 2-c.e. degree $\mathbf{d}$ such that $L[\mathbf{d}]$ doesn't have a least element.

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Spectra of Lachlan degrees

- [Fang, Wu, Yamaleev, 2013]. There exists 2-c.e. degree $\mathbf{d}$ such that $L[\mathbf{d}]$ doesn't have a minimal element.
- [Arslanov, 2000; Fang, Liu, Wu, Yamaleev, 2015]. If $\mathbf{d}$ is a properly 2-c.e. degree then $L[\mathbf{d}]$ doesn't contain a minimal pair.
- Corollary. For any properly 2-c.e. degrees $\mathbf{d}$ a minimal element in $L[\mathbf{d}]$ is the least element.
- [Fang, Liu, Wu, Yamaleev]. There exists 2-c.e. degree $\mathbf{d}$ such that $L[\mathbf{d}]$ is not bounded from below by some nonmcomputable c.e. degree.
- [Yamaleev]. For any proper 2-c.e. degree $\mathbf{d}$ its spector $L[\mathbf{d}]$ differs from $(\mathbf{0}, \mathbf{d}) \cap \mathbf{R}$.

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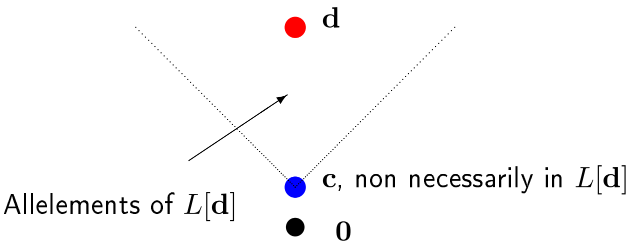
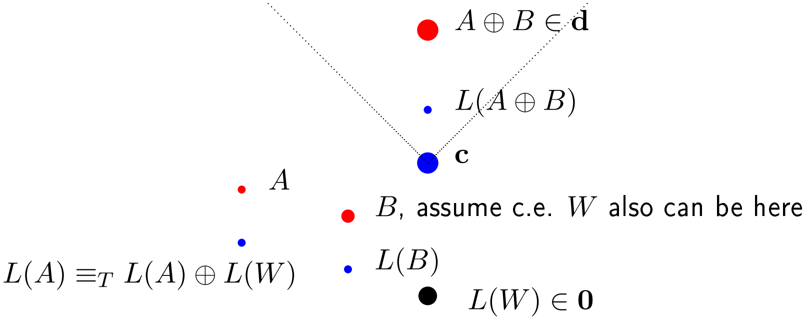


Motivation


$A \oplus B \in \mathbf{d}$
$L(A \oplus B)$
$\mathbf{c}$
A
B, assume c.e. W also can be here
$L(A) \equiv_T L(A) \oplus L(W)$
$L(B)$
$L(W) \in \mathbf{0}$

Definability of c.e. degrees in the structures with CEA

Consider 2-c.e. Turing degrees $\mathbf{D}(\leq, CEA)$.

- [Cai, Shore, 2013]. C.e. degrees definable in $\mathbf{D}(\leq, CEA)$ with Σ^0_2 formula, but not with Σ^0_1-formula.
- [Yamaleev]. For any properly 2-c.e. degree $\mathbf{d}$ its spector $L[\mathbf{d}]$ differs from the interval $(\mathbf{0}, \mathbf{d}) \cap \mathbf{R}$.
- Corollary. C.e. degrees are definable $\mathbf{D}(\leq, CEA)$ with a Π^0_1-formula.

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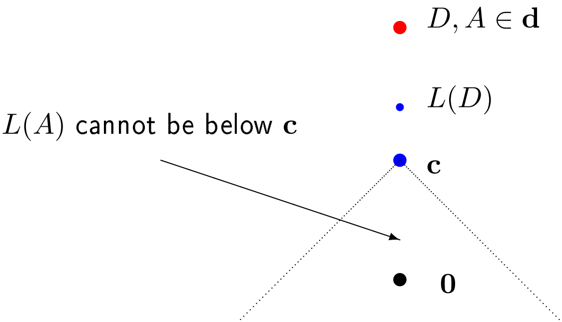
- Corollary. C.e. degrees are definable $\mathbf{D}(\leq, CEA)$ with a Π^0_1-formula.

Definability with Π^0_1 formula



Approach 4. Isolation from side.

- [Yang and Yu, 2006] Inapparently used isolation from side to show that c.e. degrees doesn't form a Σ_1-substructure of 2-c.e. degrees.
- [Cai, Slaman, and Shore, 2012] Inapparently used isolation from side to show that k-c.e. degrees doesn't form a Σ_1-substructure of n-c.e. degrees for all $k < n$
- [Wu and Yamaleev, 2012] A 2-c.e. degree $\mathbf{d}$ is *isolated from side nontrivially* if $\mathbf{d}$ is nonisolated and there exists a c.e. degree $\mathbf{a}|\mathbf{d}$ such that for all c.e. degrees $\mathbf{w}$ if $\mathbf{w} \leq \mathbf{d}$ then $\mathbf{w} \leq \mathbf{a}$.

Approach 4. The result.

Any low properly 2-c.e. Turing degree $\mathbf{d}$ is isolated from side

Theorem (Yamaleev, 2019)

For any low 2-c.e. set D with a properly 2-c.e. Turing degree there exists a c.e. set A such that $D \not\leq_T A$ and for any c.e. set $W \leq_T D$ it holds $W \leq_T A$.

- The set A can be made low
- If $D \leq_T C$ then the set A can be made below C.

Approach 4. The consequences.

- Recall from Approach 1,
  Conjecture [Arslanov, Yamaleev, 2018] The middles of double bubbles can be found below any nonzero c.e. degree, moreover it can be combined with lower cone avoidance.
- In particular, for isolation from side we bound all middles of double bubbles.

Corollary

The low c.e. degrees are definable in the partial ordering of low 2-c.e. Turing degrees

- For any low c.e. degree we can construct a definable c.e. degree below it, avoiding any lower cone
- Due to isolation from side we cannot do it for any properly low 2-c.e. degrees.

Approach 4. The definability (with parameters).

- [Welch, 1980] There exists low c.e. degrees $\mathbf{c}_1$ and $\mathbf{c}_2$ such that for any c.e. degree $\mathbf{a}$ there exists its splitting $\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{a}$ such that $\mathbf{a}_i \leq \mathbf{c}_i$ for $i = 1, 2$.
- The parameters $\mathbf{c}_1$ and $\mathbf{c}_2$ are the desired ones. Lets fix them.
- Consider a c.e. degree $\mathbf{a}$. It has the mentioned above splitting such that the both parts are below the parameters, also those parts are not isolated from side (i.e. we always can find a definable c.e. degree below them).
- Consider a properly 2-c.e. degree $\mathbf{d}$. If it doesn't have a splitting below the parameters then it is clearly proper 2-c.e. Assume it has such splitting. Then at least one part must be properly 2-c.e. Then at least one part must be isolated from side (recall that $\mathbf{c}_1$ and $\mathbf{c}_2$ are low).

Approach 4. Misc.

- Assume that for a given c.e. degrees $\mathbf{a} \not\leq \mathbf{c}$ there is a middle of bubble $\mathbf{e} < \mathbf{a}$ such that $\mathbf{e} \not\leq \mathbf{c}$. How to avoid the case when $\mathbf{c}$ could be 2-c.e.?
- Then we should update isolation from side as follows: given 2-c.e. degree $\mathbf{d}$ and c.e. degree $\mathbf{a}$ such that $\mathbf{a} \not\leq \mathbf{d}$. Then there is a c.e. degree $\mathbf{c}$ such that it covers the c.e. degrees below $\mathbf{d}$ (can include $\mathbf{d}$ as well) and $\mathbf{a} \not\leq \mathbf{c}$.

Approach 4. Backup plans.

- For a given properly 2-c.e. degree $\mathbf{d}$ do there exists c.e. degrees $\mathbf{c}$ and $\mathbf{g}$ such that one of them isolates $\mathbf{d}$ from side?
- For a given properly 2-c.e. degree $\mathbf{d}$ do there exists c.e. degrees $\mathbf{c}$ and $\mathbf{g}$ such that any c.e. degree below $\mathbf{d}$ is either below $\mathbf{c}$ or $\mathbf{g}$?
- Note that then we obtain definable degrees which are join of two middles of bubbles? Does this class coincide wth the middles of bubbles?

Turing degrees.Conclusion

- Definability with 2 parameters.
- Definability in smaller structures (low 2-c.e. degrees).
- Definability with additional predicate CEA (at least possible level).

Open questions

- Is any properly 2-c.e. degree isolated from side?
- Is any properly 2-c.e. degree pseudoisolated (by G. Wu, 2005)?
- Can c.e. degree be definable with 1 parameter in the partial ordering of 2-c.e. degrees?

Section 3

The Ershov hierarchy and the CEA hierarchy

Questions

- Given a 2-c.e. degrees. In which c.e. it can be CEA?
- Given a c.e. degree. Which 2-c.e. degrees are CEA in it?

Open question (Soare, 1994; Arlsanov, Lempp, Shore, 1996; Cooper, Li, 1998; LaForte, 2001; Arslanov, 2011)

Given low noncomputable c.e. degree $\mathbf{c}$, do there exists a properly 2-c.e. degree such that $\mathbf{d}$ is $CEA(\mathbf{c})$?

(Due to the paper of Soare and Stob, 1982)

The CEA hierarchy

- A set D is $CEA(C)$ if $C \leq_T D$ and D is Σ_1^C $(CEA = REA)$
- A degree $\mathbf{d}$ is $CEA(\mathbf{c})$ if for some $D \in \mathbf{d}$ and $C \in \mathbf{c}$ we have that D is $CEA(C)$
- A set A is n-CEA if A is $CEA(C)$ for some $(n-1)$-CEA set C
- A degree $\mathbf{d}$ is properly n-CEA if it is n-CEA, but not $(n-1)$-CEA

- C.e. degrees are just 1-CEA degrees.
- The same doesn't hold for 2-c.e. degrees.

The CEA hierarchy

Theorem (Soare, Stob, 1982)

Given noncomputable c.e. degree $\mathbf{c}$, there exists a non-c.e. degree $\mathbf{d}$ which is $CEA(\mathbf{c})$.

Theorem (Cholak, Hinman, 1994)

Given noncomputable c.e. degree $\mathbf{c}$, for all $n \geq 1$ there exists a non-n-CEA degree $\mathbf{d}$ which is $CEA(\mathbf{c})$.

Remark. In the first theorem $n = 1$, thus $\mathbf{d}$ is 2-CEA.

Enumerability relative to low c.e.degrees

In Δ^0_2-degrees:

Theorem (Soare, Stob, 1982)

Given noncomputable low c.e. degree $\mathbf{c}$, there exists a non-c.e. degree $\mathbf{d}$ which is $CEA(\mathbf{c})$

Theorem (Arslanov, Lempp, Shore, 1996)

There exists noncomplete c.e. degree $\mathbf{c}$ such that any Δ^0_2-degree, which is $CEA(\mathbf{c})$, must be c.e.

Theorem (Arslanov, LaForte, Slaman, 1998)

Given ω-c.e. degree $\mathbf{d}$, which is $CEA(\mathbf{c})$ for some c.e. degree $\mathbf{c}$. Then there exists a 2-c.e. set $D \in \mathbf{d}$ such that D is $CEA(\mathbf{c})$.

Open question (Soare, 1994; Arlsanov, Lempp, Shore, 1996; Cooper, Li, 1998; LaForte, 2001; Arslanov, 2011)

Given low noncomputable c.e. degree $\mathbf{c}$, do there exists a properly 2-c.e. degree such that $\mathbf{d}$ is $CEA(\mathbf{c})$?

The negative answer

Theorem (Arslanov, Batyrshin, Yamaleev)

There exists noncomputable low c.e. degree $\mathbf{c}$ such that any 2-c.e. degree, which is $CEA(\mathbf{c})$, must be c.e.

Corollary(Arslanov, Batyrshin, Yamaleev)

There exists noncomputable low c.e. degree $\mathbf{c}$ such that any ω-c.e. degree, which is $CEA(\mathbf{c})$, must be c.e.

Corollaries

Corollary (Arslanov, Batyrshin, Yamaleev)

There exists low c.e. degrees, which cannot be Lachlan degrees for properly 2-c.e. degrees.

Corollary (Arslanov, Batyrshin, Yamaleev)

There exists low c.e. degrees $\mathbf{b} \leq \mathbf{c}$ such that any Δ_2^0-degree, which is $CEA(\mathbf{b})$ and $> \mathbf{c}$, must be c.e.

Recall that if $\mathbf{c}$ is superlow then non-c.e. $CEA(\mathbf{c})$ degrees must be 2-c.e.

Generalization and question

Theorem (Arslanov, Batyrshin, Yamaleev)

Let $\mathcal{U}$ be a class of Δ_2^0-sets uniformly computable in $\emptyset 0'$. Do there exists a low c.e. degree such that any set from $\mathcal{U}$, which has $CEA(\mathbf{c})$ degree, must have c.e. degree?

In particular, as $\mathcal{U}$ we can take different levels of the Erhsov hierarchy.

Question (Arslanov, Batyrshin, Yamaleev)

Does the construction guarantee that the degree $CEA(\mathbf{c})$ belongs the least possible level of the Ershov hierarchy?

Question (Arslanov, Batyrshin, Yamaleev)

Given low, but non-superlow, c.e. degree $\mathbf{c}$. Do there exists $CEA(\mathbf{c})$ degree which is not of 2-c.e. degree?

Comments

- What if we try to take all Δ^0_2-sets instead of $\mathcal{U}$?
- Then we have to deal with Σ^0_2-sets as well and they ruins the lowness strategies.
- Considering incompleteness strategy we add a freedom (in particular, we can make additional copies of strategies in manner of $\mathbf{0}'''$-argument).

Acknowledgements

N.A. Bazhenov, Yu.L. Ershov, M.Kh. Faizrakhmanov, A.G. Melnikov, S.S. Ospichev, V.L. Selivanov, G. Wu, and co-authors.

Thank you for your attention!