What Could We Be, If Not Rational?

Wesley Calvert

AMS Central Section Meeting
Loyola, October 4, 2015

"If I err in my own conduct, I do not err intentionally, but from ignorance."
— Socrates

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- Bill is taller than Jenny.
- Jenny is taller than Wendy.

Problem

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Laboratory results of less than 50% success have been recorded. But it's simple to prove.

Mental Models Hypothesis (Johnson-Laird 1983, and others)

We reason by generating a mental representation to provide a workspace for inference and mental operations.

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Interpretation

4-month-olds are representing the box when they can't see it.

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Interpretation

They believe the bomb got its "impetus" to move from the plane, and loses it when it leaves the plane.

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Interpretation

Subjects solve the problem by constructing the full ordering.

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Interpretation

People do not check validity by theorem-proving, but by model checking.

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Show subjects personality descriptions, drawn from a purported pool of "engineers" and "lawyers." Tell them the pool is 70% lawyers (or engineers). Ask for the probability that a particular description is a lawyer.

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Result

It doesn't matter which you tell them is a majority; they seem to ignore this information. If you don't show them a personality description, they use the prior probability.

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Ask: Which has more days with more baby girls born than baby boys: a larger hospital, or a smaller one?

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Result

Most people think they will be about the same.

Experiment (Kahneman-Tversky 1971)

Ask an experienced quantitative psychologist: Suppose you have run an experiment on 20 subjects and have obtained a significant result which confirms your theory ($z = 2.23$, $p < .05$, two-tailed). You now have cause to run an additional group of 10 subjects. What do you think the probability is that the results will be significant, by a one-tailed test, separately for this group?

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Result

Most subjects say about .85. The truth is closer to .48.

Problem

What kind of mental models do people have that cause them to make these mistakes?

Definition

$1-$ BASIC is the (incomplete) theory in the language $(+, cdot, \leq, 0, 1)$ axiomatized by $(\forall x \forall y)$:

1. $x+1 \neq 0$
2. $(x+1=y+1) \rightarrow (x=y)$
3. $x+0=x$
4. $x+(y+1)=(x+y)+1$
5. $0+1=1$
6. $x \cdot 0=0$
7. $x \cdot (y+1)=(x \cdot y)+x$
8. $(x \leq y \wedge y \leq x) \rightarrow (x=y)$

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Theorem

If φ is a quantifier-free sentence, then $\mathrm{TA} \vdash \varphi$ if and only if $1 - \mathrm{BASIC} \vdash \varphi$.

Definition

Let Φ be a set of formulas. Then Φ-induction is the schema

$$[\varphi(0) \wedge (\forall x\ \varphi(x) \rightarrow \varphi(x+1))] \rightarrow \forall z\ \varphi(z)$$

where φ ranges over all elements of Φ.

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Definition

If Φ is the full set of formulas, then $I\Phi = PA$.

Theorem (Parikh 1971)

$I\Delta_0$ does not prove $\forall x \exists y\, [y = 2^x]$.

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Theorem

Commutative and associative properties of addition are not provable in $1 - \mathrm{BASIC}$, but they are provable in $I\mathrm{OPEN}$.

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A set S is in $NLinTime^R$ if it is decidable in time $O(n)$ on a nondeterministic multi-tape Turing machine with oracle R. We further define

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- $\Sigma_1^{lin} = NLinTime^{\emptyset}$
- $\Sigma_{n+1}^{lin} = NLinTime^{\Sigma_n^{lin}}$
- $LTH = \bigcup_i \Sigma_i^{lin}$
- $FLTH$ is the class of functions f whose graph is in LTH and so that the length of f has at most linear growth.

Theorem

A function is Σ_1-definable in $I\Delta_0$ if and only if it is in FLTH.

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There are several other complexity classes and fragments of arithmetic for which similar theorems are known.

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Given a pool of underlying assets (e.g. mortgages), with some identified (privately) as "lemons,"

Theorem (Arora–Barak–Brunnermeier–Ge, 2009)

Given a pool of underlying assets (e.g. mortgages), with some identified (privately) as "lemons," one can construct a pool of collateralized debt obligations where it is difficult (equivalent to the hidden dense subgraph problem) to detect which CDO's are overweight in lemons.

Interpretation

A "fully rational" buyer can solve the hidden dense subgraph problem and pay a fair price (or decline to buy).

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A "fully rational" buyer can solve the hidden dense subgraph problem and pay a fair price (or decline to buy). A "feasibly rational" buyer — that is, one with limited computational resources, can't do that.

High computational complexity — Strong arithmetic — Lots of mistakes

Low computational complexity — weak arithmetic — few mistakes

Question (Castelli)

Does Common Core Mathematics really ask kids to do harder things earlier?

K.CC.2 Count forward beginning from a given number within the known sequence (instead of having to begin at 1).

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Theorem (1 – BASIC)

$$\overbrace{1+1+\cdots+1}^{n+1} = \overbrace{(1+\cdots+1)}^{n} + 1$$

A-APR.2 Know and apply the remainder theorem.

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Theorem (IOPEN)

$\forall a, b \exists ! q, r \, [a = qb + r \wedge r < b]$

Problem

What about lower bounds?

A lower bound result was presented in the talk which was not ultimately correct.

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