Higher Euclidean Rings

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Euclidean Number Theory

Division Algorithm for $\mathbb{N}$

For every $0 \neq d \in \mathbb{N}$ *and every* $x \in \mathbb{N}$ *there exist numbers* $q, r \in \mathbb{N}$ *such that* $0 \leq r < d$ *and*

$$x = qd + r.$$

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Euclidean Algorithm for $\mathbb{N}$

Every pair of numbers $a, b \in \mathbb{N}$ has a greatest common divisor.

Euclidean Domains

Definition

Let R be a ring and $\varphi : R \to \mathbb{N}$ a function. We say that the pair (R, φ) is a *Euclidean ring* with *Euclidean norm* φ if for every $x, d \in R$, $d \neq 0$, there exist $q, r \in R$ such that $0 \leq \varphi(r) < \varphi(d)$ and

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Example

1. $\mathbb{Z}$, $\varphi = |\cdot|$;
2. $\mathcal{G} = \{a + bi : a, b \in \mathbb{Z}\}$, $\varphi = |\cdot|_{\mathbb{C}}$.

Transfinite Euclidean Rings

Definition

Let R be a ring, α be an ordinal, and $\varphi : R \to \alpha$. We say that the pair (R, φ) is a *Transfinite Euclidean ring* with (Transfinite) Euclidean norm φ if for every $x, d \in R$, $d \neq 0$, there exist $q, r \in R$ such that $0 \leq \varphi(r) < \varphi(d)$ and

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Theorem

Every transfinite Euclidean ring is a principal ideal domain.

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Fact

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Fact

There are noncommutative transfinite Euclidean rings of arbitrarily large ordinal ranks.

A Question

Question (Motzkin 1949, Samuel 1971)

Is there a properly Transfinite Euclidean Domain? In other words, is there a Euclidean Domain R with Euclidean Rank $> \omega$?

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Theorem (Hiblot and Nagata, 1975-77)

There is a Properly Transfinite Euclidean Domain of rank $\leq \omega^2$.

Limited technique.

A More Genreal Question

Question (Conidis, 2012)

What is the Reverse Mathematical strength of the statement (MTEF) that says "Every transfinite Euclidean ring has a minimal Transfinite Euclidean Function"?
Is there a largest possible Transfinite Euclidean Rank for all Euclidean Domains? (This generalizes Motzkin, Samuel above)

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Theorem (Conidis, 2013)

If α is terminally-admissible then ω^α is the Euclidean rank of a transfinite Euclidean domain. In particular every cardinal is the Euclidean rank of a transfinite Euclidean domain.

A Brief Overview of the Construction (Stage $s = 0$)

Let λ be terminally admissible. Let

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If $p = \prod_i p_i \in R_0$ is a product of irreducibles p_i then

$$\varphi_0(p) = \sum_i \varphi_0(p_i),$$

$\sum$ denotes the <u>*Hassenberg-Brookfield*</u> (i.e. base-ω sum) of ordinals.

Transfinite Euclidean Construction

The problem with R_0 is that it is not a Euclidean ring w.r.t. φ_0. Construct sequences

$$R_0 \subset R_1 \subset R_2 \subset \cdots$$

$$\varphi_0 \geq \varphi_1 \geq \varphi_2 \geq \cdots$$

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$$R = \bigcup R_k, \qquad \varphi = \lim_k \varphi_k.$$

R_{k+1} is a localization of R_k. R is a localization of R_0.

Constructing $R_{k+1} \supset R_k$ and $\varphi_{k+1} \leq \varphi_k$

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Choose ordinal β large not appearing in p,q and define

$$\varphi_{k+1}(p - X_\beta q) = \tau < \varphi_k(q).$$

Note that $p - X_\beta q$ is irreducible because $(p, q) = 1$ and X_β does not appear in either p or q.

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This ends the (countable) construction. Set $R = \cup_{k=1}^{\infty} R_k$, $\varphi = \lim_{k\to\infty} \varphi_k$.

Verification

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R is a (transfinite) Euclidean domain.

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Lemma

Let ψ be the unique minimal Euclidean norm on R, then for each $x \in R$ we have that

$$\varphi(x) - \psi(x) < \infty.$$

Corollary

The range of ψ (above) is ω^λ (same as φ). Hence the Euclidean rank of R is ω^λ.

Conjecture

MTEF implies Π^1_1-CA over RCA.

Thank You!