Levels of genericity and lowness for isomorphism

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Lowness and computable structure theory

A set is *low for isomorphism* if, whenever it can compute an isomorphism between two computably presented structures, there is a computable isomorphism between them.

Goal

Characterize the degrees that are low for isomorphism.

Forcing works

Being low for isomorphism can be forced: we can force functions to

- converge/diverge,
- be partial/total, and
- be surjective/not surjective.

This is enough to show that given a certain level of genericity, we can force a computable isomorphism to exist.

Interesting subclasses

Theorem (F. and Solomon)

Every 2-generic degree is low for isomorphism.

Proof.

Cohen forcing and a back-and-forth construction. □

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Mathias forcing, Martin's characterization of high degrees, and a back-and-forth construction. □

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Corollary

The degrees that are low for isomorphism do not form an ideal.

Interesting properties

We can also use forcing with perfect trees to create degrees that are low for isomorphism that have the standard properties one can get in this way.

Lowness for isomorphism and “simple” degrees

Class	Low for isomorphism	Not low for isomorphism
Δ^0_2	none	all
Δ^0_3	✓(2-gen.)	✓(Δ^0_2)
hyperimmune	✓(2-gen.)	✓(Δ^0_2)
hyperimmune free	✓(perfect trees)	✓(separating sets)
minimal	✓(perfect trees)	✓(Δ^0_2)

As a class

Theorem

The class of degrees that are low for isomorphism is comeager: it contains all 2-generics.

Theorem

The class of degrees that are low for isomorphism has measure 0: it doesn't contain any Martin-Löf randoms.

Question

Where are the boundaries of this class?

A dual notion

A degree $\mathbf{d}$ is a *degree of categoricity* if there is a computable structure $\mathcal{A}$ such that $\mathbf{d}$ can compute an isomorphism between any two computable copies of $\mathcal{A}$ and $\mathbf{d}$ is the least degree with this property.

Theorem (Fokina, Kalimullin, R. Miller)

Any degree d.c.e. in and above $0^{(n)}$ *is a degree of categoricity.*

Corollary

No degree that computes $0'$ *is low for isomorphism.*

Observation and conjecture

Degrees of categoricity are very approximable and difficult to characterize. Degrees that are low for isomorphism are far from approximable and difficult to characterize.

Furthermore, both categories are conull.

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Degrees of categoricity are very approximable and difficult to characterize. Degrees that are low for isomorphism are far from approximable and difficult to characterize.

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Question

Can we approach a characterization of both categories by studying their mutual complement?

Conjecture (F. and Solomon)

The properly 1-generic degrees are neither degrees of categoricity nor low for isomorphism.

However...

Theorem (F. and Turetsky)

There is a properly 1-generic degree that is low for isomorphism.

We construct a real G that is

- 1-generic,
- low for isomorphism, and
- not computable from a 2-generic

using a standard $\mathbf{0}''$ construction.

1-generic

Requirement:

$$\mathcal{ONE}_e\text{: } G \text{ either meets or avoids the } \Sigma^0_1 \text{ set } W_e.$$

Standard finite injury approach.

Low for isomorphism

Requirement:

$\mathcal{IM}_{\langle i,j_1,j_2 \rangle}$: If Φ_i^G is an isomorphism between $\mathcal{A}_{j_1}$ and $\mathcal{A}_{j_2}$, then $\mathcal{A}_{j_1} \cong_{\mathbf{0}} \mathcal{A}_{j_2}$.

If we see a string ρ extending our approximation g such that Φ_i^ρ appears to be a longer partial isomorphism between $\mathcal{A}_{j_1}$ and $\mathcal{A}_{j_2}$ than we had previously, we

- drop our restraint,
- extend our approximation to ρ, and
- take the infinite outcome.

Not computable from a 2-generic

Requirement:

$\mathcal{TWO}_i$: There is a Σ^0_2 set X_i such that if $\Phi_i^Y = G$, then Y neither meets nor avoids X_i.

Subrequirements:

$\mathcal{TWO}_{\langle i,\tau\rangle}$: If there is a $Y \succ \tau$ with $\Phi_i^Y = G$, then Y does not meet X_i and there is a string $\rho \succ \tau$ with $\rho \in X_i$.

We spread these subrequirements out along our priority tree and identify candidates for the set X_i as we satisfy them.

One subrequirement

Suppose we have a finite approximation g to G and we want to satisfy $\mathcal{TWO}_{\langle i,\tau \rangle}$. We reserve the bit b at the next position for our use and require that $G(b) = 0$.

Now we look for a $\rho \succeq \tau$ such that $\Phi_i^\rho \succeq g^\frown 0$. If we find one, we choose it as our ρ and flip $G(b)$ to 1.

In the end...

We define the TP as usual, and for each i, we define X_i to be the set of ρ such that

- there is a $\mathcal{TWO}_{\langle i,\tau\rangle}$-node $\nu \leq TP$ with g_ν defined,
- $\tau \preceq \rho$, and
- $\Phi_i^\rho \succeq g_\nu\frown 0$.

Since the construction is $\mathbf{0}''$, this will be a Σ_2^0 set.

Verification

Claim

Y does not meet or avoid X_i for any Y such that $\Phi_i^Y = G$.

Y avoids X_i: Fix a $\tau \prec Y$ at which this happens, and let σ be the corresponding string on the TP. By definition, there is no $\rho \succeq \tau$ with $\Phi_i^\rho \succeq g_\sigma{}^\frown 0$, so $\Phi_i^Y(|g_\sigma|) \neq 0$. But if there's no such ρ, then $g_\sigma{}^\frown 0 \prec G$.

Y meets X_i due to a node not on the TP: Find a $\rho \prec Y$ where this happens, and choose the node ν witnessing $\rho \in X_i$. We have $\Phi_i^\rho \succeq g_\nu{}^\frown 0$, but since ν isn't on the TP, g_ν is not an initial segment of G and $\Phi_i^Y \neq G$.

Y meets X_i due to a node on the TP: We'll have $g_\nu{}^\frown 1 \prec G$, guaranteeing that $\Phi_i^Y \neq G$.

Thank you!