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Computable Categoricity and Scott Families

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Latest results joint with E. Fokina and D. Turetsky.

Computable and Relative Computable Categoricity

Let A be a *computable* structure.

- A is *computably categorical* if for all computable $B \cong A$, there is a computable isomorphism f from A onto B.

- A is *relatively computably categorical* if for all $B \cong A$, there is an isomorphism f from A onto B, which is computable relative to the atomic diagram of B.

- A is relatively computably categorical $\Rightarrow$ A is computably categorical

- $(\mathbb{Q}, <)$ is computably categorical.

- $(\omega, <)$ is not computably categorical.

- A computable random graph is computably categorical.

- (R. Miller 2005) No computable tree $(T, \prec)$ of infinite height is computably categorical.

- The field $\mathbb{Q}$ is computably categorical.

- (R. Miller-Shoutens 2013; Kudinov-Lvov)
  There is a computable computably categorical field of
  infinite transcendence degree.

Syntactic Approach to Categoricity

Let A be a *countable* structure.

- A *Scott family for* A is a set Φ of $L_{\omega_1\omega}$ formulas, with a fixed finite tuple of parameters $\overline{c}$ in A, such that:

1. Each tuple $\overline{a}$ in A satisfies some $\psi(\overline{c}, \overline{x}) \in \Phi$, and

2. If $\overline{a}$, $\overline{b}$ are tuples in A (of the same length) satisfying the *same* formula $\psi(\overline{c}, \overline{x}) \in \Phi$, then there is an *automorphism* of A taking $\overline{a}$ to $\overline{b}$.

- (Goncharov 1975)

  A computable structure A is relatively computably categorical *iff* A has a c.e. Scott family of (finitary) existential formulas *iff* A has a c.e. Scott family of computable $\mathbf{\Sigma_1}$ formulas.

  Computable $\mathbf{\Sigma_1}$ formula: $\bigvee_{i \in I} \exists \overline{u_i}\, \theta_i(\overline{x}, \overline{u_i})$, I is c.e. and θ_i's are quantifier-free.

- (Ash-Knight-Manasse-Slaman 1989, Chisholm 1990)

  A computable structure A is *relatively* $\mathbf{\Delta^0_\alpha}$-categorical *iff* A has a $\mathbf{\Sigma^0_\alpha}$ Scott family of computable $\mathbf{\Sigma_\alpha}$ formulas *iff* A has a c.e. Scott family of computable $\mathbf{\Sigma_\alpha}$ formulas.

Structures Computably Categorical but Not Relatively

- (Goncharov 1977)
  There is a computable structure (in fact, a rigid graph) that is computably categorical, but *not* relatively computably categorical.

- (Hirschfeldt-Khoussainov-Shore-Slinko 2002)
  There are computable computably categorical, but *not* relatively computably categorical: partial orders, lattices, 2-step nilpotent groups, commutative semigroups, integral domains.

- (Hirschfeldt-Kramer-Miller-Shlapentokh 2015)
  There is a computable computably categorical algebraic field, which is *not* relatively computably categorical.

Computable Categoricity $\Rightarrow$Relative Computable Categoricity

- (Goncharov-Dzgoev 1980, Remmel 1981)
  A computable linear ordering A is computably categorical *iff*
  A has only finitely many successor pairs *iff*
  A is relatively computably categorical.

- (LaRoche 1977, Goncharov-Dzgoev 1980, Remmel 1981)
  A computable Boolean algebra B is computably categorical *iff*
  B has finitely many atoms *iff*
  B is relatively computably categorical.

- (Miller-Shlapentokh 2015)
  A computable algebraic field F with a splitting algorithm
  is computably categorical *iff* the *orbit relation*,
  $\{(a, b) \in F^2 : (\exists h \in Aut(F))[h(a) = b]\}$, is computable *iff*
  F is relatively computably categorical.

- F has a *splitting algorithm* if it is decidable
  which polynomials in $F[x]$ are irreducible.

- (Goncharov 1980, Smith 1981)
  A computable Abelian p-group *G* is computably categorical *iff*
  G is isomorphic to:
  (1) $\bigoplus_{\alpha} \mathbb{Z}(p^{\infty}) \oplus F$, where $\alpha \leq \omega$ and F is finite, *or*
  (2) $\bigoplus_{n} \mathbb{Z}(p^{\infty}) \oplus F \oplus \bigoplus_{\omega} \mathbb{Z}(p^{k})$, where $n, k \in \omega$ and F is finite
  iff G is relatively computably categorical.

- (Calvert-Cenzer-Harizanov-Morozov 2006)
  A computable equivalence structure $A = (D, E)$ is
  computably categorical *iff* either
  (1) A finitely many finite equivalence classes, or
  (2) A has finitely many infinite classes, a finite bound on the size of finite classes, and exactly one finite k with infinitely many classes of size k.
  iff A is relatively computably categorical.

- (Lempp-McCoy-Miller-Solomon 2005)
  Every computable computably categorical tree of finite height is relatively computably categorical.

- An injection structure $A = (D, f)$, where $f : D \to D$ is a $1-1$ function.
  For $a \in D$, the *orbit* of a is:

$$\mathcal{O}_f(a) = \{b \in D : (\exists n \in \omega)[f^n(a) = b \ \vee \ f^n(b) = a]\}$$

- (Cenzer-Harizanov-Remmel 2014)
  A computable injection structure A is computably categorical *iff*
  A has finitely many infinite orbits *iff*
  A is relatively computably categorical.

Extra Decidability and Categoricity

- (Goncharov 1975)
  Assume that A is 2-*decidable*. If A is computably categorical, then A is relatively computably categorical.

  A is n-*decidable* if $\mathbf{\Sigma}^0_n$ elementary diagram of A is computable.

- (Ash 1987)
  Let $\alpha > 1$ be a computable ordinal.
  Under some additional decidability on A, if A is $\mathbf{\Delta}^0_\alpha$-categorical, then A is relatively $\mathbf{\Delta}^0_\alpha$-categorical.

- (Kudinov 1996)
  There is a 1-*decidable* structure that is computably categorical, but *not* relatively computably categorical.

- (Cholak-Goncharov-Khoussainov-Shore 1999)
  There is a computable computably categorical structure A such that for every $a \in A$, the expanded structure (A, a) is *not* computably categorical.

- (T. Millar 1986)
  If a computably categorical structure A is 1-decidable, then any expansion of A by finitely many constants remains computably categorical.

- (Downey-Kach-Lempp-Turetsky 2013)
  Any 1-decidable computably categorical structure is relatively Δ_2^0-categorical.

Fraïssé limits

- The *age* of a structure A is the class of all finitely generated structures that can be embedded in A.

- A structure A is *ultrahomogeneous* if every isomorphism between finitely generated substructures of A extends to an automorphism of A.

- A structure A is a *Fraïssé limit* of a class of finitely generated structures $\mathbb{K}$ if A is countable, ultrahomogeneous, and has age $\mathbb{K}$. (A structure A is a Fraïssé limit if for some class $\mathbb{K}$, A is the Fraïssé limit of $\mathbb{K}$.)

- (Fraïssé) Fraïssé limit of a class of finitely generated structures is unique up to isomorphism.

- (Fokina-Harizanov-Turetsky 2015)
  There is a 1-decidable structure F that is a Fraïssé limit and is computably categorical, but not relatively computably categorical. Moreover, the language for such F can be finite or relational.

- Let A be a computable structure which is a Fraïssé limit. Then A is relatively $\mathbf{\Delta}_2^0$-categorical.

- If the language of a Fraïssé limit A is finite and relational, then A is relatively computably categorical.

Non-Relatively Δ^0_α-Categorical Structures

- (Goncharov-Harizanov-Knight-McCoy-Miller-Solomon 2005)
  For every computable *successor* ordinal $\alpha = \beta + 1$, there is a computable structure that is Δ^0_α-categorical, but *not* relatively Δ^0_α-categorical.

- (Chisholm-Fokina-Goncharov-Harizanov-Knight-Quinn 2009)
  For every computable *limit* ordinal α, there is a computable structure that is Δ^0_α-categorical, but *not* relatively Δ^0_α-categorical.

- (Downey-Kach-Lempp-Lewis-Montalbán-Turetsky 2015)
  For every computable ordinal α, there is a computably categorical structure that is *not* relatively Δ^0_α-categorical.

Δ^0_2-*Categoricity of Structures from Natural Classes*

- (McCoy 2003)
  A computable Boolean algebra $\mathcal{B}$ is *relatively* Δ^0_2-*categorical iff* $\mathcal{B}$ can be expressed as a finite direct sum of subalgebras

$$\mathcal{C}_0 \oplus \cdots \oplus \mathcal{C}_k$$

  where each $\mathcal{C}_k$ is either atomless, an atom, or a $\mathbf{1}$-atom.

- (Bazhenov 2014; Harris)
  Every computable Δ^0_2-categorical Boolean algebra is relatively Δ^0_2-categorical.

- (Cenzer-Harizanov-Remmel 2014)
  A computable injection structure A is $\mathbf{\Delta}_2^0$-categorical *iff*
  A has finitely many orbits of type ω or finitely many orbits of type Z *iff*
  A is relatively $\mathbf{\Delta}_2^0$-categorical.

- (Calvert-Cenzer-Harizanov-Morozov 2006)
  A computable equivalence structure A is relatively $\mathbf{\Delta}_2^0$-categorical *iff*:
  (1) A has finitely many infinite equivalence classes, or
  (2) A has a finite bound on the size of finite equivalence classes.

- (Kach-Turetsky 2009)
  There is a computable $\mathbf{\Delta}_2^0$-categorical equivalence structure M,
  which is *not* relatively $\mathbf{\Delta}_2^0$-categorical.

- (Fokina-Harizanov-Turetsky 2015)
  There is a Δ^0_2-categorical tree of finite (or infinite) height, which is not relatively Δ^0_2-categorical.

- (Calvert-Cenzer-Harizanov-Morozov 2009)
  A computable Abelian p-group G is *relatively* Δ^0_2*-categorical iff*

  (1) G is isomorphic to $\bigoplus_{\alpha} \mathbb{Z}(p^\infty) \oplus H$, where $\alpha \leq \omega$ and H has finite period; *or*

  (2) all elements in G are of finite height
  (equivalently, reduced with $\lambda(G) \leq \omega$).

- The *period* of a group H is $max\{o(h) : h \in H\}$ if finite, and ∞ otherwise.

- (Fokina-Harizanov-Turetsky 2015)
  There is a computable Δ^0_2-categorical Abelian p-group G, which is *not* relatively Δ^0_2-categorical.

- A *homogenous, completely decomposable,* abelian group is a group of the form $\bigoplus_{i\in I} H$, where H is a subgroup of $(\mathbb{Q}, +)$.
  Let $\mathcal{H}$ be the class of such groups.

- $G \in \mathcal{H}$ is (relatively) computably categorical *iff* G is of finite rank.

- For P, a set of primes, $Q^{(P)}$ is the subgroup $(\mathbb{Q}, +)$ generated by $\{\frac{1}{p^k} \ : \ p \in P \wedge k \in \omega\}$.

- (Downey-Melnikov 2014)
  Let a computable $G \in \mathcal{H}$ be of infinite rank. Then G is Δ^0_2-categorical *iff* G is isomorphic to $\bigoplus_\omega Q^{(P)}$, where P is c.e. and the set (Primes $- P$) is semi-low.

- A set $S \subseteq \omega$ is *semi-low* if the set
  $H_S = \{e : W_e \cap S \neq \emptyset\}$ is computable from $\emptyset'$.

- (Fokina-Harizanov-Turetsky 2015)
  (i) A computable $G \in \mathcal{H}$ of infinite rank is relatively Δ^0_2-categorical *iff* G is isomorphic to $\bigoplus_\omega Q^{(P)}$ where P is a *computable* set of primes.

  (ii) There is a computable $G \in \mathcal{H}$, which is Δ^0_2-categorical, but not relatively Δ^0_2-categorical.

- (McCoy 2003)
  A computable linear order A is relatively Δ^0_2-categorical *iff*
  A is a sum of finitely many intervals, each of type

$$m, \omega, \omega^*, \mathbb{Z}, \text{ or } n \cdot \eta,$$

  so that each interval of type $n \cdot \eta$ has a supremum and an infimum.

- *Open Problem:* Is there is a computable Δ^0_2-categorical linear order, which is *not* relatively Δ^0_2-categorical?

- *Open Problem*: Is every Δ^1_1-categorical structure relatively Δ^1_1-categorical?

THANK YOU!