Uniform reduction and reverse mathematics
Preliminary report

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Motivation

Goal: Explore the relationship between uniform (also called Weihrauch) reducibility and results in reverse mathematics.

Observation: Some reducibility results and reverse mathematics results have proofs with significant common content.

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For example, in [1], Gura, Hirst, and Mummert prove:

$$\mathsf{RCA}_0 \vdash \mathsf{FC1} \leftrightarrow \mathsf{FC3} \quad \text{and} \quad \mathsf{FC1} \equiv_{\mathrm{sW}} \mathsf{FC3}$$

where

FC1 says: every infinite graph in which every connected component is finite has a sequence of canonical indices of different components.

FC3 says: every infinite graph in which every connected component is finite has an infinite totally disconnect set.

Motivation

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Observation: Some reducibility results and reverse mathematics results have proofs with significant common content.

We can reduce duplication in our arguments if we can prove single results that have both desired consequences as immediate corollaries.

Formalizing *sW* reduction

One characterization of *sW* reduction is to consider *problems*:

> The problem P is a sentence $\forall X \exists Y\ p(X, Y)$, where $p(X, Y)$ is a formula of second order arithmetic.
>
> If $p(X_P, Y_P)$, we say X_P is an instance of the problem P and Y_P is a solution of X_P.

In this setting $Q \leqslant_{sW} P$ means there are computable functionals ψ and ϕ such that

$$\begin{array}{ccc} & \psi & \\ X_Q & \longrightarrow & X_P \\ \downarrow & & \downarrow \\ Y_Q & \longleftarrow & Y_P \\ & \phi & \end{array}$$

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In this setting $Q \leqslant_{sW} P$ means there are computable functionals ψ and ϕ of type $1 \to 1$ such that

$$\begin{array}{ccc} & \psi & \\ X_Q & \longrightarrow & X_P \\ \downarrow & & \downarrow \\ Y_Q & \longleftarrow & Y_P \\ & \phi & \end{array}$$

Kohlenbach's axioms

Kohlenbach [3] presents axioms for reverse mathematics in higher types.

- RCA_0^ω consists of $\widehat{\text{E-HA}}^\omega_\restriction$ plus law of the exclude middle plus $\text{QF-AC}^{1,0}$:

$$\forall X \exists y\, A(X, y) \;\rightarrow\; \exists Y \forall X\, A(X, Y(X))$$

  for A quantifier free.

- $\widehat{\text{E-HA}}^\omega_\restriction$ is intuitionistic arithmetic in all finite types. (See §3.4 of Kohlenbach [4]).

- $\widehat{\text{E-HA}}^\omega_\restriction$ includes combinators allowing λ-abstraction.

Formalizing sW reduction

Since the functionals defining *sW* reduction are of finite type, statements about their existence can be formulated in higher order reverse mathematics.

Since the type structure ECF is a model of RCA_0^ω that contains only computable functionals (see [3]), if $\mathrm{RCA}_0^\omega \vdash Q \leqslant_{sW} P$, then $Q \leqslant_{sW} P$.

By composition of functionals,

$$\mathrm{RCA}_0^\omega \vdash Q \leqslant_{sW} P \rightarrow (P \rightarrow Q \wedge \widehat{P} \rightarrow \widehat{Q})$$

where $\widehat{P}$ is the infinite parallelization of P.

By Proposition 3.1 of Kohlenbach [3]:

If $\mathrm{RCA}_0^\omega \vdash \theta$ then $\mathrm{RCA}_0 \vdash \theta$.

A sample problem

Goal: Prove $\mathrm{RCA}_0^\omega \vdash \widehat{\mathrm{LPO}} \leqslant_{sW} \mathrm{RAN}$.

$\widehat{\mathrm{LPO}}$ is $\forall \langle p_n \rangle \, \exists g \, (g(i) = 1 \leftrightarrow \exists t \, p_i(t) = 0)$

So g selects those i such that 0 is in the range of p_i.
Infinite parallelization of the limited principle of omniscience.

RAN is "Every injective function has a range."

$$\forall f \, \exists \chi_f \, \forall y \, (\chi_f(y) = 1 \leftrightarrow \exists t \, f(t) = y)$$

$\widehat{\text{LPO}} \leqslant_{sW}$ RAN: Construction of ϕ in RCA_0^ω

Given $\langle p_n \rangle$ for $\widehat{\text{LPO}}$, define an injection f by $f((i,j)) = k$ if and only if the following formula (denoted $\theta(\langle p_n \rangle, (i,j), k)$) holds:

$$(k = 2i + 1 \wedge p_i(j) = 0 \wedge \forall t < j \; p_i(t) \neq 0) \vee$$
$$(k = 2(i,j) \wedge (p_i(j) \neq 0 \vee \exists t < j \; p_i(t) = 0))$$

Note that $2i + 1 \in \text{RAN}(f)$ if and only if $\exists t \; p_i(t) = 0$, so

$$\chi_{\text{RAN}(f)}(2i+1) = \begin{cases} 0 & \text{if } \forall t \; p_i(t) \neq 0 \\ 1 & \text{if } \exists t \; p_i(t) = 0 \end{cases}$$

which is the solution to the instance $\langle p_n \rangle$ of $\widehat{\text{LPO}}$.

Define ϕ by $\phi(\chi_{\text{RAN}(f)}) = \chi_{\text{RAN}(f)}(2i+1)$.

$\widehat{\text{LPO}} \leqslant_{sW}$ RAN: Construction of ψ in RCA_0^ω

Working in RCA_0^ω, we need to prove the existence of the functional ψ mapping $\langle p_n \rangle$ to f (as defined on the previous slide).

Our main tool is $\text{QF-AC}^{1,0}$: $\forall X \exists y\, A(X, y) \to \exists Y \forall X\, A(X, Y(X))$

$\theta(\langle p_n \rangle, (i, j), k)$ is Σ_0^0 and $\forall(\langle p_n \rangle, (i, j)) \exists k\, \theta(\langle p_n \rangle, (i, j), k)$, so $\text{QF-AC}^{1,0}$ proves the existence of a functional F such that $\theta(\langle p_n \rangle, (i, j), F(\langle p_n \rangle, (i, j)))$.

F maps $(\langle p_n \rangle, (i, j))$ to $f((i, j))$.

Thus f is $\lambda(i, j).F((\langle p_n \rangle, (i, j)))$
and $\psi = \lambda \langle p_n \rangle.[\lambda(i, j).F((\langle p_n \rangle, (i, j)))]$.

Summarizing the demonstration problem:

We showed $\mathsf{RCA}_0^\omega \vdash \widehat{\mathsf{LPO}} \leqslant_{sW} \mathsf{RAN}$. Similar techniques can be used to prove $\mathsf{RCA}_0^\omega \vdash \mathsf{RAN} \leqslant_{sW} \widehat{\mathsf{LPO}}$, so

$$\mathsf{RCA}_0^\omega \vdash \widehat{\mathsf{LPO}} \equiv_{sW} \mathsf{RAN}.$$

Consequently,

Because every functional in the ECF model of RCA_0^ω is computable,

$$\widehat{\mathsf{LPO}} \equiv_{sW} \mathsf{RAN}$$

By Kohlenbach's conservation result,

$$\mathsf{RCA}_0 \vdash \widehat{\mathsf{LPO}} \leftrightarrow \mathsf{RAN} \quad \text{and} \quad \mathsf{RCA}_0 \vdash \widehat{\widehat{\mathsf{LPO}}} \leftrightarrow \widehat{\mathsf{RAN}}$$

Because $\mathsf{RCA}_0^\omega \vdash \widehat{P} \equiv_{sW} \widehat{\widehat{P}}$,

$$\mathsf{RCA}_0 \vdash \mathsf{ACA}_0 \leftrightarrow \widehat{\mathsf{LPO}} \leftrightarrow \widehat{\widehat{\mathsf{LPO}}} \leftrightarrow \mathsf{RAN} \leftrightarrow \widehat{\mathsf{RAN}}$$

Questions

- How unfaithful is this formalization of sW reduction? Find good examples where $P \leqslant_{sW} Q$ but $\mathsf{RCA}_0^\omega \nvdash P \leqslant_{sW} Q$. In particular, what about statements that are equivalent to the pigeonhole principle or to Σ^0_2 induction?
- If $\mathsf{RCA}_0^\omega \nvdash P \leqslant_{sW} Q$, then we can view the formalization of $P \leqslant_{sW} Q$ as a "functional existence axiom" which is not provable in RCA_0^ω. What is the logical strength of these functional existence axioms? How are they related to the $\rightarrow$ operator on Weihrauch problems?
- What about other reducibilities?

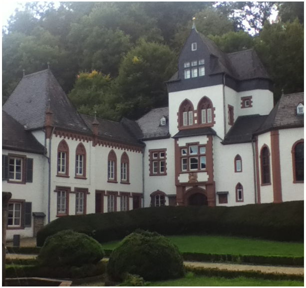
A week ago, Schloss Dagstuhl - Leibniz-Zentrum für Informatik held Dagstuhl Seminar 15392: *Measuring the Complexity of Computational Content: Weihrauch Reducibility and Reverse Analysis.*

Organizers: V. Brattka, A. Kawamura, A. Marcone, A. Pauly

Associated bibliography:

http://cca-net.de/publications/weibib.php

Summaries of talks will eventually appear in Dagstuhl Reports



Some references

[1] Jeffry L. Hirst, Carl Mummert, and Kirill Gura, *On the existence of a connected component of a graph*, Computability **4** (2015), 103–117. DOI 10.3233/COM-150039.
Draft at: http://mathsci2.appstate.edu/ jlh/bib/pdf/hmg-graph-final.pdf.

[2] Jeffry L. Hirst and Carl Mummert, *Reverse mathematics and uniformity in proofs without excluded middle*, Notre Dame J. Form. Log. **52** (2011), no. 2, 149–162, DOI 10.1215/00294527-1306163.
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[3] Ulrich Kohlenbach, *Higher order reverse mathematics*, Reverse mathematics 2001, Lect. Notes Log., vol. 21, Assoc. Symbol. Logic, La Jolla, CA, 2005, pp. 281–295.

[4] U. Kohlenbach, *Applied proof theory: proof interpretations and their use in mathematics*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2008.