Computability strength of the field of real numbers

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Definition (Schweber)

Let $\mathcal{A}$ and $\mathcal{B}$ be structures, potentially uncountable. Then $\mathcal{A} \leq_w^* \mathcal{B}$ if, after a forcing collapse that causes $\mathcal{A}$ and $\mathcal{B}$ to both become countable, every copy of $\mathcal{B}$ computes a copy of $\mathcal{A}$.

Under reasonable hypotheses, this reducibility does not depend on the forcing used.

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In practice, very little set theory is involved: most proofs can be written by just imagining that A and B were countable, and seeing what would happen.

Our Structures

- $\mathcal{R}_{exp} = (\mathbb{R}, +, \cdot, <, exp)$
- $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$
- $\mathcal{R}_{+} = (\mathbb{R}, +, <)$
- $\mathcal{R}_{\mathbb{Q}} = (\mathbb{R},\ \text{constants “}q\text{”}_{(q \in \mathbb{Q})}, <)$
- $\mathcal{B} = (\omega^{\omega},\ \text{predicates “}f(n) = m\text{”}) \equiv_{w}^{*} (\mathbb{R},\ \text{binary expansion})$
- $\mathcal{W} = (P(\omega),\ \text{predicates “}n \in \ \text{”})$

Observation (Knight, Montalban, Schweber; IKS)

$\mathcal{W} \leq_w^* \mathcal{B} \leq_w^* \mathcal{R}_{\mathbb{Q}} \leq_w^* \mathcal{R}_{+} \leq_w^* \mathcal{R} \leq_w^* \mathcal{R}_{exp}$

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The Relations

Theorem (Igusa, Knight, Schwebber)

$\mathcal{W} <^*_w \mathcal{B} \equiv^*_w \mathcal{R}_{\mathbb{Q}} \equiv^*_w \mathcal{R}_+ \equiv^*_w \mathcal{R} \equiv^*_w \mathcal{R}_{exp}$

Theorem (Downey, Greenberg, Miller)

$\mathcal{W} <^*_w \mathcal{B} \quad \equiv^*_w \quad \mathcal{R}_+ \equiv^*_w \mathcal{R}$

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- Every countable Scott set S has a real closed field realizing exactly the types in S that is equicomputable with S. (Macintyre and Marker)
- if S is $\mathcal{W}$, then this real closed field, $\widetilde{\mathcal{R}}$, is a recursively saturated extension of $\mathcal{R}$, so $\mathcal{R}$ is the residue field of $\widetilde{\mathcal{R}}$

Theorem (Igusa, Knight)

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Proof Ingredients

Effective structure theory!

- A property is computable Σ_2 definable in a structure if and only if it is Σ_2^0 in every copy of the structure.
  (Ash, Knight, Manasse, Slaman)
- If $k \leq_w K$, then $FT(K)$, the set of finite elements of K in transcendental Dedekind cuts, is computable Σ_2 definable in K.
  (A direct $0'$-style construction)
- If K is recursively saturated, then $FT(K)$ is not computable Σ_2 definable in K.
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Proving that $FT(K)$ is not computable Σ_2 definable in K:

- If $FT(K)$ were definable via a $\exists\forall-$ formula, then each $\forall-$ part would have to omit all algebraic Dedekind cuts.
- Recursive saturation + compactness of first order logic means that it must omit a neighborhood of that cut, and that it must do so at a finite stage.
- We then dive into that neighborhood, but not that cut, dodging both the $\forall-$ formula and the algebraic cut.
- We then repeat this infinitely many times to produce a transcendental element that does not satisfy any of the $\forall-$ formulas.
- Recursive saturation guarantees that this element exists in K.

Theorem (DGM)

If I is a countable Scott ideal, then to list all the functions in I, you must be able to compute a function dominating all of them, but you can list all the sets in I without doing so.

(A forcing proof.)

Theorem (DGM)

If I is a countable Scott ideal, then from a list of all the functions in I, you can compute the field of reals whose Turing degrees are in I.

(Uses quantifier elimination and decidability of $Th(RCF)$.)

(Note, in both of these, I is the set of all Turing degrees in $\mathcal{W}$ or equivalently $\mathcal{B}$.)

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Properties from $\mathcal{R}_{\mathbb{Q}}$:

- Ability to code $Th(\mathbb{R}, +, \cdot, exp)$ using a parameter.
- Ability to code infinite paths through a tree in a Π^0_1 manner.
- Ability to code open rational boxes.
- Ability to list all the reals, and compute which boxes they are in.

Properties from $\mathcal{R}_{exp}$:

- o-minimal.
- Any copy of $\mathcal{R}_{\mathbb{Q}}$ has a unique expansion to $\mathcal{L}(\mathbb{R}_{exp})$.

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- Choose a parameter coding *Th*$(\mathbb{R}, +, \cdot, exp)$.
- Using this parameter, we find an algebraicity basis for $\mathcal{R}_{exp}$ in $\mathcal{R}_{\mathbb{Q}}$ in a Δ^0_2 way.

  (Uses the fact that a tuple is algebraically independent if and only if every formula that is true about it is true on a rational box around it.)
- Using Dedekind approximations to a basis, together with a parameter for the theory, can build a copy of $\mathcal{R}_{exp}$.

  Add finite injury to the construction because we have a Δ^0_2 approximation to the basis.

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Going Further

The function exp can be replaced by any other f such that $\mathcal{R}_f$ is o-minimal.

Using restrictions to compact intervals, we can get any analytic f. (van den Dries, Gabrielov)

Classically, $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if f is computable from a parameter.

In our context, this only shows that $\mathcal{R}_{\mathbb{Q}} \equiv^*_w \mathcal{R}_{\mathbb{Q},f}$.

Question

Is $\mathcal{R} \equiv^*_w \mathcal{R}_f$ for an arbitrary continuous f?

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End

Thank you