Computability strength of the field of real numbers

Gregory Igusa*, Julia Knight, Noah Schweber

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Definition (Schweber)

Let \mathcal{A} and \mathcal{B} be structures, potentially uncountable. Then $\mathcal{A} \leq_w^* \mathcal{B}$ if, after a forcing collapse that causes \mathcal{A} and \mathcal{B} to both become countable, every copy of \mathcal{B} computes a copy of \mathcal{A} .

Under reasonable hypotheses, this reducibility does not depend on the forcing used.

This agrees with \leq_w on countable structures

In practice, very little set theory is involved: most proofs can be written by just imagining that *A* and *B* were countable, and seeing what would happen.

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Observation (Knight, Montalban, Schweber; IKS)

 $\mathcal{W} \leq^*_w \mathcal{B} \leq^*_w \mathcal{R}_Q \leq^*_w \mathcal{R}_+ \leq^*_w \mathcal{R} \leq^*_w \mathcal{R}_{exp}$

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- Every countable Scott set *S* has a real closed field realizing exactly the types in *S* that is equicomputable with *S*. (Macintyre and Marker)
- if S is W, then this real closed field, R, is a recursively saturated extension of R, so R is the residue field of R

Theorem (Igusa, Knight)

Let K be a countable recursively saturated real closed field with residue field k. Then $k \not\leq_w K$.

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Effective structure theory!

- A property is computable Σ₂ definable in a structure if and only if it is Σ₂⁰ in every copy of the structure. (Ash, Knight, Manasse, Slaman)
- If k ≤_w K, then FT(K), the set of finite elements of K in transcendental Dedekind cuts, is computable Σ₂ definable in K.
 (A direct 0'-style construction)
- If K is recursively saturated, then FT(K) is not computable Σ₂ definable in K.
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Proving that FT(K) is not computable Σ_2 definable in K:

- If *FT*(*K*) were definable via a ∃∀− formula, then each ∀− part would have to omit all algebraic Dedekind cuts.
- Recursive saturation + compactness of first order logic means that it must omit a neighborhood of that cut, and that it must do so at a finite stage.
- We then dive into that neighborhood, but not that cut, dodging both the ∀− formula and the algebraic cut.
- We then repeat this infinitely many times to produce a transcendental element that does not satisfy any of the ∀− formulas.
- Recursive saturation guarantees that this element exists in *K*.

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Theorem (DGM)

If I is a countable Scott ideal, then to list all the functions in I, you must be able to compute a function dominating all of them, but you can list all the sets in I without doing so.

(A forcing proof.)

Theorem (DGM)

If I is a countable Scott ideal, then from a list of all the functions in I, you can compute the field of reals whose Turing degrees are in I.

(Uses quantifier elimination and decidability of Th(RCF).)

(Note, in both of these, *I* is the set of all Turing degrees in W or equivalently \mathcal{B} .)

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$\mathcal{R}_{\mathbb{Q}} \equiv^*_{w} \mathcal{R}_{exp}$ Igusa, Knight, Schwebber

Properties from $\mathcal{R}_{\mathbb{Q}}$:

- Ability to code $Th(\mathbb{R}, +, \cdot, exp)$ using a parameter.
- Ability to code infinite paths through a tree in a Π_1^0 manner.
- Ability to code open rational boxes.
- Ability to list all the reals, and compute which boxes they are in.

Properties from \mathcal{R}_{exp} :

- o-minimal.
- Any copy of $\mathcal{R}_{\mathbb{Q}}$ has a unique expansion to $\mathcal{L}(\mathbb{R}_{exp})$.

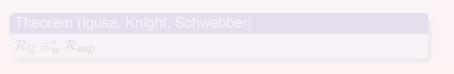
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• Choose a parameter coding $Th(\mathbb{R}, +, \cdot, exp)$.

• Using this parameter, we find an algebraicity basis for \mathcal{R}_{exp} in $\mathcal{R}_{\mathbb{Q}}$ in a Δ_2^0 way.

(Uses the fact that a tuple is algebraically independent if and only if every formula that is true about it is true on a rational box around it.)

 Using Dedekind approximations to a basis, together with a parameter for the theory, can build a copy of *R_{exp}*.

Add finite injury to the construction because we have a $\Delta_2^{\rm V}$ approximation to the basis.

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Using restrictions to compact intervals, we can get any analytic *f*. (van den Dries, Gabrielov)

Classically, $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if f is computable from a parameter.

In our context, this only shows that $\mathcal{R}_{\mathbb{Q}} \equiv_{w}^{*} \mathcal{R}_{\mathbb{Q},f}$.

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C. J. Ash, J. F. Knight, M. Manasse, and T. Slaman, "Generic copies of countable structures", *Ann. Pure Appl. Logic*, vol. 42(1989), pp. 195-205.

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G. Igusa and J. F. Knight, "Comparing two versions of the reals", submitted.

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Thank you

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