Subclasses of the K-trivial degrees

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A brief introduction to K-triviality

We use K to denote prefix-free (Kolmogorov) complexity.

Definition

An infinite binary sequence $A \in 2^\omega$ is K-trivial if

$$(\exists c)(\forall n)\ K(A \upharpoonright n) \leq K(n) + c.$$

In other words, a K-trivial sequence has *minimal* initial segment prefix-free complexity (up to a constant); its initial segments are no more complex that those of the zero sequence.

Basic Facts

- Every computable sequence is K-trivial.
- [Chaitin 1970's] Every K-trivial sequence is $\leq_T \emptyset'$.
- [Solovay 1975] There is a non-computable K-trivial sequence.

A brief introduction to K-triviality

The study of K-trivial sequences stalled until the early 2000's, when many equivalent characterizations were found.

Theorem (Nies 2005; Hirschfeldt, Nies, Stephan 2007)

Let $A \in 2^\omega$. The following are equivalent:

1. A is K-trivial: $(\exists c)(\forall n)\ K(A \upharpoonright n) \leq K(n) + c$,
2. A is low for K: $(\exists c)(\forall \sigma)\ K^A(\sigma) \geq K(\sigma) - c$,
3. A is low for randomness: every ML-random sequence is ML-random relative to A,
4. A is a base for randomness: there is an $X \geq_T A$ that is ML-random relative to A.

- Each of these properties was introduced years before they were all proved to be equivalent.
- Many other characterizations have now been given.

A brief introduction to K-triviality

Along with these characterizations came a greater understanding of the class of K-trivial sequences:

More Facts

- Every K-trivial sequence is low (i.e., $A' \leq_T \emptyset'$).
- The K-trivial sequences form an ideal in the Turing degrees: they are closed downward under Turing reducibility and closed under join.
- Each K-trivial sequence is computable from a K-trivial c.e. set.

More recent work on K-triviality has focused on the relationship between the K-trivial sequences and the Martin-Löf random sequences in the Turing degrees. In particular:

The ML-covering question (Stephan 2004)

If A is K-trivial, is there a ML-random $X \geq_T A$ such that $X \not\geq_T \emptyset'$?

ML-covering and variants

The ML-covering question (Stephan 2004)

If A is K-trivial, is there a ML-random $X \geq_T A$ such that $X \not\geq_T \emptyset'$?

This was eventually answered in positively, and in fact, it was proved that there is a Martin-Löf random $X <_T \emptyset'$ that computes every K-trivial sequence [Day, M. 2015] + [Bienvenu, Greenberg, Kučera, Nies, Turetsky 2015].

But consider another variant of the ML-covering question:

Question (M. and Nies 2006)

If A is K-trivial, is there a Martin-Löf random sequence $X = X_1 \oplus X_2$ such that A is computable from both X_1 and X_2?

- If $X = X_1 \oplus X_2$ is ML-random, then at least one of X_1 and X_2 fails to compute $\emptyset'$, so this question is a strengthening of the ML-covering question.
- However, [Bienvenu, Greenberg, Kučera, Nies, Turetsky 2015] answered this question negatively.

1/2-bases

Definition

A sequence $A \in 2^\omega$ is a 1/2-base if there is a ML-random sequence $X = X_1 \oplus X_2$ such that A is computable from both X_1 and X_2.

Facts

- Every 1/2-base is K-trivial.

Proof.

Assume that A is computable from both halves of the ML-random sequence $X = X_1 \oplus X_2$. Since X_1 is ML-random relative to X_2, it is also ML-relative relative to $A \leq_T X_2$. Thus $A \leq_T X_1$ is a base for randomness. Therefore, A is K-trivial. $\square$

- As was mentioned above, not every K-trivial is a 1/2-base.
- However, not all 1/2-bases are computable [Kučera 1986].

So the 1/2-bases form a proper subclass of the K-trivial sequences.

1/2-bases

Let Ω be any left-c.e. ML-random (e.g., the halting probability of a universal prefix-free machine).

Theorem (Greenberg, M., Nies)

Let $A \in 2^\omega$. The following are equivalent:

1. A is a 1/2-base,
2. A obeys the cost function $\mathbf{c}(m, s) = \sqrt{\Omega_s - \Omega_m}$,
3. A is computable from both Ω_1 and Ω_2, where $\Omega = \Omega_1 \oplus \Omega_2$,
4. [with Turetsky] A is K-trivial and computable from Ω_1.

- It follows that the 1/2-bases form an ideal. (The downward closure was clear, but not the closure under join.)
- We also proved that each 1/2-base is computable from a c.e. 1/2-base.
- ($1 \iff 4$) implies that Ω_1 and Ω_2 compute the same c.e. sets.

Generalizing to k/n-bases

Definition

Let $k < n$. A sequence $A \in 2^\omega$ is a k/n-base if there is a Martin-Löf random sequence $Z = Z_1 \oplus \cdots \oplus Z_n$ such that A is computable from the join of any k of the n parts of Z.

Theorem (Greenberg, M., Nies)

Let $A \in 2^\omega$. The following are equivalent:

1. A is a k/n-base,
2. A obeys $\mathbf{c}(m, s) = (\Omega_s - \Omega_m)^{k/n}$,
3. A is a k/n-base as witnessed by Ω,
4. [with Turetsky] A is K-trivial and computable from some k/n part of Ω.

As before, the k/n-bases form an ideal in the Turing degrees that is generated by its c.e. elements.

A dense hierarchy of ideals

We now see that the 1/2-bases are the same as the 2/4-bases. (One direction is not at all obvious from the definition!)

In general, we can talk of p-bases for $p \in (0,1)$ rational. Let $\mathcal{B}_p$ be the ideal of p-bases.

Facts

- If $q < p$, then $\mathcal{B}_q \subsetneq \mathcal{B}_p$.
- There is a smart p-base, i.e., a p-base A such that every ML-random that computes A computes every p-base.
- A smart p-base cannot be a q-base for any $q < p$, so $\bigcup_{q<p} \mathcal{B}_q \subsetneq \mathcal{B}_p$. It is also the case that $\mathcal{B}_p \subsetneq \bigcap_{q>p} \mathcal{B}_q$.
- $\bigcap_{p>0} \mathcal{B}_p$ is the ideal consisting of $1/\omega$-bases: A is a $1/\omega$-base if there is a ML-random sequence $Z = Z_0 \oplus Z_1 \oplus \cdots$ such that $A \leq_T Z_i$ for every $i \in \omega$. [Greenberg and Turetsky] This ideal is also generated by its c.e. members.

We will discuss the ideal $\bigcup_{p<1} \mathcal{B}_p$ below.

More generality, but no new ideals

Fact

Assume that A is a 3/6-base as witnessed by the ML-random sequence $Z = Z_1 \oplus Z_2 \oplus Z_3 \oplus Z_4 \oplus Z_5 \oplus Z_6$. In addition, assume that $A \leq_T Z_1 \oplus Z_2$. Then A is a 3/7-base (as witnessed by a different random sequence).

Furthermore, this is tight; if A is a 3/7-base, then there is such a Z.

- In general, arbitrary families of projections (along the coordinate axes!) do not give us new subideals of the K-trivial sets.
- Proved using a generalization of the Loomis–Whitney inequality from geometry. Gives an upper bound for the measure of a region in terms of the measures of its projections.
- Using a linear program, we can find an optimal bound for a given family of projections. This tells us for which p the family characterizes the p-bases, as in the example above.

Corollaries of the general result

Definition

Let $k < n$. A sequence $A \in 2^\omega$ is a degenerate k/n-base if there is a ML-random sequence Z witnessing that A is a k/n-base and such that A is computable from the join of some $k-1$ of the n parts of Z.

Degenerate k/n-bases are always p-bases for a smaller p:

Proposition

Let $p = \max\left\{\dfrac{k}{n+1}, \dfrac{k-1}{n-1}\right\}$. A set is a degenerate k/n-base if and only if it is a p-base.

Definition

Let $k < n$. A sequence $A \in 2^\omega$ is a cyclic k/n-base if there is a ML-random sequence Z such that A is computable from all n of the "cyclic joins" of k of the n parts of Z.

Proposition. A set is a cyclic k/n-base if and only if it is a k/n-base.

Robust computability

We finish by discussing another proper subclass of the K-trivials, from work of Hirschfeldt, Jockusch, Kuyper, and Schupp.

Definitions

- $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$, i.e., the symmetric difference of the sets X and Y.
- If $\lim_{n\to\infty} \frac{|(X \triangle Y) \cap \{0, \ldots, n-1\}|}{n} = 0$, then we say that Y is a coarse description of X. In other words, Y is an imperfect copy of X, where the imperfections have asymptotic density 0.
- A is robustly computable from X if A is computable from every coarse description of X.

Theorem (Hirschfeldt, Jockusch, Kuyper, and Schupp)

If A is robustly computable from some ML-random sequence, then A is K-trivial. *In fact, A is an $(n-1)/n$-base for some n.*

Robust computability

Theorem (Hirschfeldt, Jockusch, Kuyper, and Schupp)

There is a non-computable A that is robustly computable from Ω.
Not every K-trivial is robustly computable from some random.

So the sequences that are robustly computable from some random form a *proper* subclass of the K-trivials (contained in $\bigcup_{p<1} \mathcal{B}_p$).

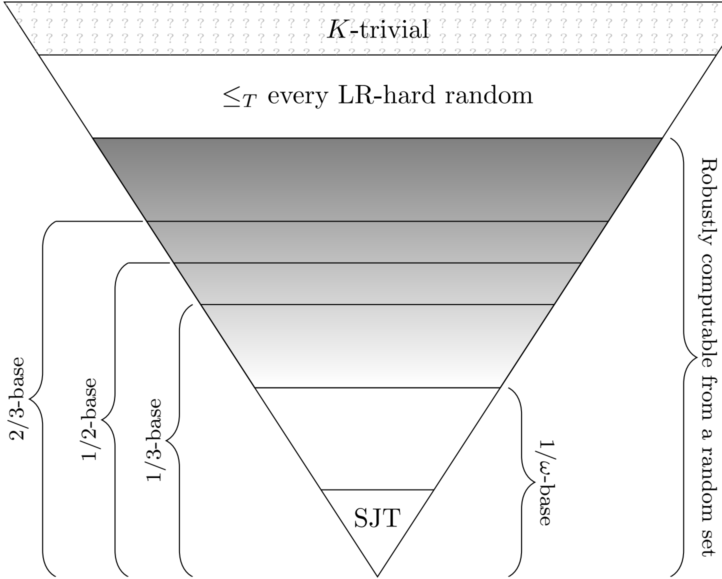
Theorem (Greenberg, M., Nies)

Let $A \in 2^\omega$. The following are equivalent:

1. A is robustly computable from some ML-random sequence,
2. A is a p-base for some $p < 1$ (i.e., $A \in \bigcup_{p<1} \mathcal{B}_p$),
3. A is robustly computable from Ω.

So the sequences that are robustly computable from some random form a proper *subideal* of the K-trivials, generated by its c.e. elements.

Subideals of the K-trivial degrees



Thank You!