Coarse Reducibility and Algorithmic Randomness

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Main Reference

I will mainly discuss the following paper:

"Coarse reducibility and algorithmic randomness" by Denis Hirschfeldt, Carl Jockusch, Rutger Kuyper, and Paul Schupp (to appear in J. Symbolic Logic).

After the paper was submitted, significant improvements of some of the results were obtained by Gregory Igusa (private communication), based on his work with Peter Cholak and Damir Dzhafarov.

(Asymptotic) density of subsets of ω

Ideally, an algorithm to solve a problem will always answer correctly. However, this is not always achievable. We will consider algorithms which give correct answers with asymptotic density 1.

Let $A \subseteq \omega$, $n \in \omega$.

(I) $\rho_n(A) = |A \cap \{0, 1, \ldots, n-1\}|/n$ (density of A up to n)

(II) $\rho(A) = \lim_n \rho_n(A)$, provided the limit exists. (Density of A.)

For example, the multiples of k have density $1/k$, and the squares have density 0.

Generic Computability

Generic computability was defined by Kapovich, Myasnikov, Schupp and Shpilrain (2003) and applied (with time bounds) to analyze decision problems in group theory.

Definition

Let $A \subseteq \omega$. Then A is *generically computable* if there is a partial computable function φ such that:

1. $\varphi(n) = A(n)$ for all n in the domain of φ
2. The domain of φ has density 1

Coarse Computability

We now consider algorithms which always give an answer but may err on a set of density 0.

Definition

A set A is *coarsely computable* if there is a (total) computable function f such that $\{n : f(n) = A(n)\}$ has density 1.

Jockusch and Schupp (2012) showed that neither of generic computability and coarse computability implies the other.
In this talk I will focus on coarse computability and the corresponding reducibility concepts.

Definition

A set D is a *coarse description* of a set A if $\{n : D(n) = A(n)\}$ has density 1.

Clearly, A is coarsely computable if and only if A has a computable coarse description.

THE OPERATOR $\mathcal{I}$

The following operator was used by Jockusch and Schupp to show that every nonzero Turing degree contains a set which is neither generically computable nor coarsely computable.

DEFINITION

Define $\mathcal{I} : 2^\omega \to 2^\omega$ as follows:

$$\mathcal{I}(A) = \cup_{n \in A} I_n$$

where I_n is the interval $[n!, (n+1)!)$.

PROPOSITION

(J-S) Let $A \subseteq \omega$.

- $A \equiv_T \mathcal{I}(A)$
- *A is computable from every coarse description D of $\mathcal{I}(A)$.*

Recall that $\mathcal{I}(A) = \cup_{n \in A} I_n$. To prove that A is computable from every coarse description D of $\mathcal{I}(A)$, use majority vote on each interval. Specifically:

$$A =^* \{n : |D \cap I_n| \geq \frac{1}{2}|I_n|\}$$

where $=^*$ denotes equality mod finite sets.

COROLLARY

(J-S) Every nonzero Turing degree contains a set which is not coarsely computable.

Coarse reducibilities

Definition

1. The set B is *nonuniformly coarsely reducible to A* (written $B \leq_{nc} A$) if every coarse description D of A computes some coarse description E of B.
2. The set B is *uniformly coarsely reducible* to A (written $B \leq_{uc} A$) if there is a Turing functional Φ such that, for every coarse description D of A, Φ^D is a coarse description of B.

It is obvious that both $\leq_{nc}$ and $\leq_{uc}$ are reflexive and transitive, and hence there are corresponding degree notions. In each, there is a least degree consisting of the coarsely computable sets.

Example

For every set X, $X \leq_{uc} \mathcal{I}(X)$.

Many other reducibilities of this sort have been considered by Dzhafarov and Igusa, replacing coarse descriptions by other kinds of partial descriptions.

An embedding

Recall that $\mathcal{I}(A) = \cup_{n \in A}[n!, (n+1)!)$.
Then $\mathcal{I}$ induces an embedding of the Turing degrees in the nonuniform coarse degrees:

Proposition

Let B and A be any subsets of ω. Then

$$B \leq_T A \iff \mathcal{I}(B) \leq_{nc} \mathcal{I}(A)$$

Proof.

This follows at once from the fact that, for every X, $\mathcal{I}(X) \leq_T X$ and every coarse description of $\mathcal{I}(X)$ computes X. $\square$

Embedding the Turing degrees into the uniform coarse degrees

The mapping $\mathcal{I}$ induces an embedding of the Turing degrees into the nonuniform coarse degrees.

However, this embedding does not appear to work for embedding the Turing degrees into the *uniform* coarse degrees, because from a coarse description of $\mathcal{I}(A)$, one may uniformly compute only a set *almost* equal to A, and this does not suffice to uniformly compute a set $B \leq_T A$.

THE OPERATOR $\mathcal{E}$

Let

$$A \times B = \{\langle a, b\rangle : a \in A \;\&\; b \in B\}$$

Define

$$\mathcal{E}(A) = \mathcal{I}(A \times \omega)$$

The following was observed independently by us and Dzhafarov and Igusa.

THEOREM

For any sets $A, B \subseteq \omega$,

$$B \leq_T A \iff \mathcal{E}(B) \leq_{uc} \mathcal{E}(A)$$

Hence, the operator $\mathcal{E}$ induces an embedding of the Turing degrees into the uniform coarse degrees.

The difference between uniform and nonuniform coarse reducibility

We showed that uniform and nonuniform coarse reducibility are distinct. Then our result was improved by Igusa, who gave examples of c.e. sets where these reducibilities differ.

Definition

A set A is *autoreducible* if there is a Turing functional Φ such that, for all n, $A(n) = \Phi^{A \cup \{n\}}(n)$.

There are c.e. sets which are *not* autoreducible.

Remark

For all X, $\mathcal{E}(X) \leq_{nc} \mathcal{I}(X)$ and $\mathcal{I}(X) \leq_{uc} \mathcal{E}(X)$.

Theorem (Igusa)

If $\mathcal{E}(X) \leq_{uc} \mathcal{I}(X)$ then X is autoreducible.

This result is proved using various other reducibilities introduced by Dzhafarov and Igusa. We do not see how to prove it directly.

Corollary

There are c.e. sets A and B such that $B \leq_{nc} A$ but $B \not\leq_{uc} A$.

Proof.

Let $A = \mathcal{E}(X)$ and $B = \mathcal{I}(X)$, where X is a c.e. set which is not autoreducible. □

QUASIMINIMALITY

A set X is *quasiminimal* for $\leq_{nc}$ if X is not coarsely computable and, for all noncomputable $A \subseteq \omega$, $\mathcal{E}(A) \not\leq_{nc} X$.

Thus, for X not coarsely computable, X is quasiminimal for $\leq_{nc}$ iff no noncomputable set is computable from every coarse description of X.

Quasiminimality is defined analogously for $\leq_{uc}$.

Of course, no degree of a quasiminimal set is in the range of the embedding for either $\leq_{nc}$ or $\leq_{uc}$.

If X is random in some sense, we expect very little information to be recoverable from every coarse description of X, and so it is reasonable to conjecture that X is quasiminimal.

Theorem

If X is 1-random and A is computable from every coarse description of X, then A is K-trivial.

The main tool in the proof is a kind of compactness theorem. The *lower density* of a set Y is $\liminf_n \rho_n(Y)$.

Theorem

Let $X, A \subseteq \omega$. Suppose that for every real $\epsilon > 0$ there is a set $D_\epsilon \not\geq_T A$ such that $\{n : D_\epsilon(n) = X(n)\}$ has lower density at least $1 - \epsilon$. Then X has a coarse description $D \not\geq_T A$.

A set X is *weakly n-random* if X belongs to no Π^0_n set $P \subseteq 2^\omega$ of measure 0.

COROLLARY

Every weakly 2-random set is quasiminimal for $\leq_{nc}$ and hence also for $\leq_{uc}$.

PROOF.

Every K-trivial set is Δ^0_2, and no weakly 2-random set can compute any noncomputable Δ^0_2 set. $\square$

Contrasting results

Theorem

Suppose that $X \leq_T 0'$ is 1-random. Then there is a promptly simple set A such that if D and X agree on a set of lower density greater than $\frac{3}{4}$, then $A \leq_T D$.

Corollary

If $X \leq_T 0'$ is 1-random, then X is not quasiminimal for $\leq_{nc}$.

The following contrasting result of Cholak and Igusa answers a question we raised.

Theorem

(Cholak and Igusa) If X is 1-random, then X is quasiminimal for $\leq_{uc}$.

This is proved by considering other reducibilities.

Many characterizations are known for the K-trivial sets. We have shown that every set computable from every coarse description of a 1-random set is K-trivial. However, this does not give a characterization of K-triviality:

Theorem

There is a K-trivial set A such that every 1-random set X has a coarse description $D \not\geq_T A$.

Minimal pairs in the Turing degrees (Review)

Definition

In any partially ordered structure with least element $\mathbf{0}$, two elements $\mathbf{a}, \mathbf{b}$ form a *minimal pair* if neither $\mathbf{a}$ nor $\mathbf{b}$ is $\mathbf{0}$, but the infimum of $\mathbf{a}$ and $\mathbf{b}$ is $\mathbf{0}$.

Theorem

(Kučera) If $X, Y \leq_T 0'$ are 1-random, then the Turing degrees of X and Y do not form a minimal pair.

Proof.

X and Y each compute promptly simple sets, and no two promptly simple sets form a minimal pair. □

Theorem

If $X \oplus Y$ is weakly 2-random, then the Turing degrees of X and Y form a minimal pair in the Turing degrees.

Minimal pairs in the uniform and nonuniform coarse degrees

The situation is similar to that for Turing degrees, but "one jump up".

Theorem

Suppose that $X, Y \leq_T 0''$ are 2-random. Then the nonuniform coarse degrees of X and Y do not form a minimal pair in the nonuniform coarse degrees.

The proof uses the Sacks jump inversion theorem and other results.

Question

Does the above result hold for the uniform coarse degrees?

Theorem

If $X \oplus Y$ is weakly 3-random, then the nonuniform coarse degrees of X and Y form a minimal pair. (The corresponding result for uniform coarse degrees follows immediately.)