Strong jump inversion

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Computable structures

We consider structures with universe a subset of ω, and having a computable language.

We identify a structure $\mathcal{A}$ with its atomic diagram $D(\mathcal{A})$, and we identify this with a subset of ω.

Strong jump inversion, strong n^{th} jump inversion

Definition. Let $\mathcal{A}$ be a structure.

1. $\mathcal{A}$ *admits strong jump inversion* if whenever $\mathcal{A}$ has a copy that is low over X, there is an X-computable copy.

2. $\mathcal{A}$ *admits strong n^{th} jump inversion* if whenever $\mathcal{A}$ has a copy that is *low$_n$* over X, there is an X-computable copy.

Boolean algebras

Downey-Jockusch. Let $\mathcal{A}$ be a Boolean algebra. If X' computes a copy of $\mathcal{A}$ with the added predicate $atom(x)$, then X computes a copy of $\mathcal{A}$.

If $\mathcal{A}$ has a copy that is low over X, then X' computes the predicate $atom(x)$. Hence, all Boolean algebras admit strong jump inversion.

Thurber, K-Stob. All Boolean algebras admit strong n^{th} jump inversion, for $n = 2, 3, 4$.

Lerman and Schmerl

Proposition (Lerman-Schmerl). Suppose T is an $\aleph_0$-categorical theory such that $T \cap \exists_{n+1}$ is Σ^0_n for all n. Let $\mathcal{A}$ be a countable model of T. Then T admits strong n^{th} jump inversion for all n.

Proof.

The *n-quantifier-diagram* of $\mathcal{A}$ is the set of sentences in the complete diagram that have at most n alternations of quantifiers in prenex normal form. Suppose $\mathcal{A}$ is low_n over X. Then the n-quantifier-diagram is Δ^0_{n+1} over X ($X^{(n)}$-computable).

Lerman and Schmerl showed that if there is a copy for which the $(k+1)$-quantifier diagram is Y'-computable, then there is a copy for which the k-quantifier diagram is Y-computable. Working our way down, we get a copy whose atomic diagram is X-computable. $\square$

Further examples

Proposition. The following structures admit strong jump inversion.

1. equivalence structures with infinitely many infinite classes.
2. Abelian p-groups of length ω such that the divisible part has infinite dimension.

Not all structures admit strong jump inversion

Example 1. Take a low completion T of PA. There is a model $\mathcal{A}$ whose complete diagram is computable in T. Let $\mathcal{B}$ be the expansion of $\mathcal{A}$ with all finitary definable relations. The jump of $\mathcal{B}$ is Δ_2^0. Since $\mathcal{A}$ is necessarily non-standard, Tennenbaum's Theorem says that there is no computable copy.

Example 2. Jockusch and Soare showed that there is a low linear ordering with no computable copy. This ordering does not admit strong jump inversion.

Goal, and preliminary findings

Goal. Look for general conditions that guarantee strong jump inversion, and that explain as many as possible of the known examples.

In much of computable structure theory, when we are interested in a specific structure, we use infinitary formulas. Here it seems helpful to use elementary first order formulas.

Preliminary findings. Structures that admit strong jump inversion seem to have some saturation properties. Moreover, the 1-quantifier-types have nice names/descriptions.

B_1-formulas and types

We make more precise what we mean by 1-quantifier-types.

1. A B_1*-formula* is a finite Boolean combination of existential formulas.

2. A B_1*-type* is the set of B_1-formulas true of a tuple $\bar{a}$ in a structure $\mathcal{A}$.

We may also consider B_n-formulas, B_n-types.

1. In a Boolean algebra, the B_1-type of a tuple $\bar{a}$ gives the sizes of atoms in the finite algebra generated by $\bar{a}$.
2. In a linear ordering, the B_1-type of a tuple $\bar{a}$ gives the ordering and the sizes of intervals.

Weak 1-saturation

Definition. A structure $\mathcal{A}$ is *weakly* 1-*saturated* provided that if $p(\bar{u})$ is the B_1-type of a tuple $\bar{a}$, and $q(\bar{u}, x)$ is a B_1-type generated by formulas of $p(\bar{u})$ and existential formulas, then $q(\bar{a}, x)$ is realized in $\mathcal{A}$.

Examples.

1. A Boolean algebra is weakly 1-saturated if it has no 1-atoms.
2. A linear ordering is weakly 1-saturated if every infinite interval splits into two infinite intervals (with endpoints in the ordering).

Lemma. Let $p(\bar{u})$ be a B_1-type. Suppose $q(\bar{u}, x)$ is a B_1-type that is generated by formulas of $p(\bar{u})$ and existential formulas. Then $q(\bar{u}, x)$ is consistent with all extensions of $p(\bar{u})$ to a complete type in variables $\bar{u}$.

Enumerations and labelings

Definition. Let S be a countable family of sets. An *enumeration* of S is a set R of pairs (i, k) such that S is the family of sets $R_i = \{k : (i, k) \in R\}$. If $A = R_i$, we say that i is an R-*index* for A.

Definition Let S be a set of B_1-types including all those realized in $\mathcal{A}$. Let R be an enumeration of S. An *R-labeling of $\mathcal{A}$* is a function taking each tuple $\bar{a}$ in $\mathcal{A}$ to an R-index for the B_1-type of $\bar{a}$.

First general result, first result of Frolov

Proposition. Suppose $\mathcal{A}$ is weakly 1-saturated. Let R be a computable enumeration of a set S of B_1-types including all those realized in $\mathcal{A}$. If $\mathcal{A}$ has an R-labeling that is Δ^0_2 relative to X, then $\mathcal{A}$ has an X-computable copy $\mathcal{B}$. Moreover, there is an isomorphism from $\mathcal{B}$ onto $\mathcal{A}$ that is Δ^0_2 relative to X.

Frolov proved that certain kinds of linear orderings admit strong jump inversion. The proposition above immediately gives one of his results.

Theorem A (Frolov). Let $\mathcal{L}$ be a linear ordering such that every element lies on a maximal discrete set that is finite. Suppose there is a finite bound on the sizes of these sets. Then $\mathcal{L}$ admits strong jump inversion.

If $\mathcal{L}$ is low, over X then Δ^0_2 relative to X, we can label the tuples, giving the sizes of the intervals.

Second result of Frolov

Theorem B (Frolov). Let $\mathcal{L}$ be a linear ordering in which every element lies on a maximal discrete set that is finite or of type ω, ω^* or $\zeta = \omega^* + \omega$. Suppose that the quotient $\mathcal{L}/_\sim$ (with one point for each maximal discrete set) has order type η. Finally, suppose that every infinite interval has arbitrarily large finite successor chains. Then $\mathcal{L}$ admits strong jump inversion.

For simplicity, we ignore X and show that if $\mathcal{L}$ has a low copy, then it has a computable copy.

Lemma (Frolov). Suppose $\mathcal{L}$ is low. There is a copy $\mathcal{A}$, Δ^0_2 over X, with the successor relation and with a Δ^0_2 function $f : \mathcal{A} \to \mathbb{Q}$ inducing an isomorphism from $\mathcal{A}/_\sim$ to $\mathbb{Q}$.

This $\mathcal{A}$ has a Δ^0_2 labeling giving the sizes of intervals. Then by the first general proposition, we get a computable copy.

Second general result, with some undefined terms

Theorem. Let $\mathcal{A}$ be a structure for a finite language. Let S be a set of B_1-types including those realized in $\mathcal{A}$. Let G be a computable set of B_1-formulas such that S has "finite character" with respect to G. Let R be a computable enumeration of S that "exhibits" G. Suppose $\mathcal{A}$ satisfies "weak G-saturation" and "buffering". If $\mathcal{A}$ is low over X, then there is a copy $\mathcal{B}$ of $\mathcal{A}$ with an R-labeling that is Δ^0_2 over X. Moreover, there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$ that is Δ^0_3 relative to X.

This applies to the linear orderings in Frolov's Theorem B, and to a few other examples. In the next two slides, we define "finite character", "exhibiting G", "weak G-saturation", and "buffering".

Fnite character, exhibiting special formulas

Let S be a set of B_1-types including those realized in $\mathcal{A}$, with a computable enumeration R.

1. **Finite character**. There is a computable set of formulas G such that each B_1-type in S is generated by a finite set of G-formulas plus existential formulas. Each type contains only finitely many G-formulas, and two types that include the same G-formulas are equal.

2. **Exhibiting** G. The computable enumeration R has the feature that given i, we can compute the set of G-formulas in the type R_i.

Weak G-saturation, buffering

3. **Weak G-saturation**. For a tuple $\bar{a}$, if R_i extends the type of $\bar{a}$ and $G \cap R_i = \alpha$ and the formulas of α are all true of $\bar{a}$ and some $\bar{b}$, then R_i is the type of $\bar{a}$ and some $\bar{c}$.

4. **Buffering**. For a tuple $\bar{c}$ and a B_1-type $p(\bar{c}, x)$ realized in $\mathcal{A}$, we can find, by a procedure that is Δ^0_2 relative to $\mathcal{A}$, a B_1-type $q(\bar{c}, \bar{u})$, also realized in $\mathcal{A}$, and a quantifier-free type $\delta(\bar{c}, \bar{u}, x)$, such that if $\bar{z}$ satisfies $q(\bar{c}, \bar{u})$, then any B_1-type $p'(\bar{c}, x, \bar{v})$, extending $p(\bar{c}, x)$ and realized in $\mathcal{A}$, is realized by some $a, \bar{a}'$ such that $\delta(\bar{c}, \bar{z}, a)$ holds, and if $\bar{z}$ satisfies $q(\bar{c}, \bar{u})$ and a satisfies $\delta(\bar{c}, \bar{z}, x)$ plus the G-formulas of $p(\bar{c}, x)$, then a must satisfy $p(\bar{c}, x)$.

Second jumps

We have an analog of the first general result.

Proposition. Let $\mathcal{A}$ be "1-saturated" and "weakly 2-saturated". Suppose R^1 is a computable enumeration of a set of B_1-types including those realized in $\mathcal{A}$, and R^2 is a computable enumeration of a set of B_2-types including those realized in $\mathcal{A}$. If there is copy of $\mathcal{A}$ with an R^2-labeling that is Δ^0_3 relative to X, then there is a copy with an R_1-labeling that is Δ^0_2 relative to X. Moreover, there is an isomorphism that is Δ^0_3 relative to X.

By the first general result, $\mathcal{A}$ has an X-computable copy.

Application. Let $\mathcal{B}$ be a Boolean algebra with no 1-atoms and no $I(\omega + \eta)$. If $\mathcal{B}$ has a copy that is low_2 relative to X, then there is an X-computable copy. Moreover, if $\mathcal{B}$ is such a Boolean algebra, low_2 over X, there is an X-computable copy, with a Δ^0_3 isomorphism.

Problems

1. Improve the second general result.
2. Show that not every low_5 Boolean algebra has a computable copy.