A measure of uniformity

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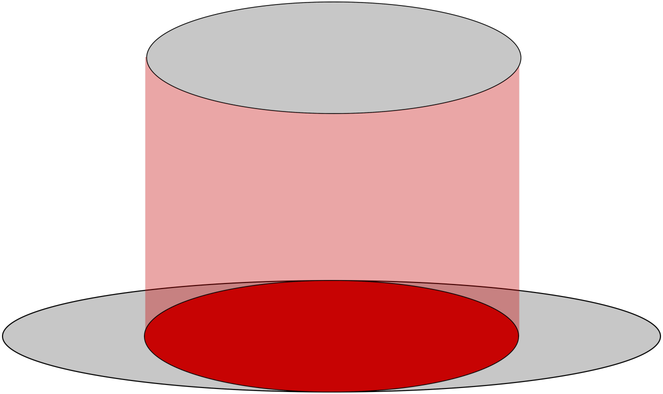
4 October 2015

Medvedev reducibility

Definition. Let $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$. Then we say that $\mathcal{A}$ *Medvedev reduces* to $\mathcal{B}$ ($\mathcal{A} \leq_{\mathcal{M}} \mathcal{B}$) if there is a single Turing functional Φ such that $\Phi(\mathcal{B}) \subseteq \mathcal{A}$.

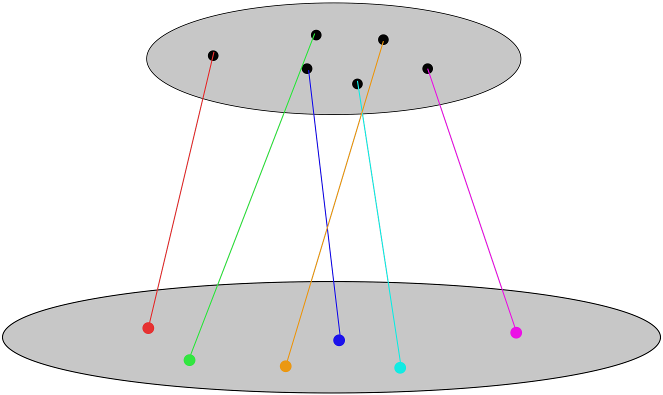
Medvedev reducibility



Muchnik reducibility

Definition. Let $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$. Then we say that $\mathcal{A}$ *Muchnik reduces* to $\mathcal{B}$ ($\mathcal{A} \leq_w \mathcal{B}$) if and only if for every $g \in \mathcal{B}$ there exists $f \in \mathcal{A}$ with $f \leq_T g$.

Muchnik reducibility



Where uniformity fails

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Proof. Majority vote. □

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Proof. A kind of majority vote. □

Intermediate degree structures

Definition. Let $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$ and let $n \in \omega$. Then we say that $\mathcal{A}$ *n-uniformly reduces to* $\mathcal{B}$ (notation: $\mathcal{A} \leq_n \mathcal{B}$) if there exists a sequence $\mathcal{V}_0, \mathcal{V}_1, \dots$ of uniformly Π^0_n sets with $\mathcal{B} \subseteq \bigcup_{i\in\omega} \mathcal{V}_i$ and a uniformly computable sequence $e_0, e_1, \dots$ such that for every $i \in \omega$ and every $f \in \mathcal{B} \cap \mathcal{V}_i$ we have $\Phi_{e_i}(f) \in \mathcal{A}$.

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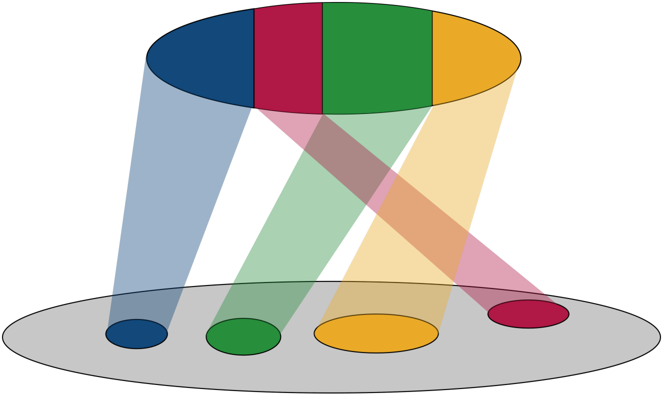

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For $n = 1$, this structure was also studied by Higuchi and Kihara, although in a different setting and with a different (but equivalent) definition.

Some elementary results

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Proposition. *For every $n \in \omega \cup \{\infty\}$, $\mathcal{M}_n$ is a distributive lattice. In fact, it is even a Brouwer algebra: there is an operation $\rightarrow_n$ such that*

$$\mathcal{A} \oplus \mathcal{C} \geq_n \mathcal{B} \Leftrightarrow \mathcal{C} \geq_n \mathcal{A} \rightarrow_n \mathcal{B}.$$

Going back and forth

Proposition. *Let $n, m \in \omega \cup \{\infty\}$ with $n \leq m$. Then the natural surjection from $\mathcal{M}_n$ onto $\mathcal{M}_m$ (induced by the identity map) preserves $\oplus$ and $\otimes$, but not necessarily $\rightarrow$.*

Theorem. ($m = 0$, $n = \infty$: Sorbi; $m = 0$,$n = 1$: Higuchi and Kihara) *Let $n, m \in \omega \cup \{\infty\}$ with $n \leq m$. Then there is an embedding of $\mathcal{M}_n$ into $\mathcal{M}_m$ preserving $\oplus$ and $\rightarrow$, but not necessarily $\otimes$.*

$$\mathcal{M}_0 \overset{\hookleftarrow}{\twoheadrightarrow} \mathcal{M}_1 \overset{\hookleftarrow}{\twoheadrightarrow} \mathcal{M}_2 \overset{\hookleftarrow}{\twoheadrightarrow} \cdots \overset{\hookleftarrow}{\twoheadrightarrow} \mathcal{M}_\infty$$

Levels of uniformity

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Proposition. (Higuchi and Kihara) *Let* $\mathcal{A} \leq_w \mathcal{B}$ *be such that* $\mathcal{A}$ *is* Σ^0_{n+1}. *Then the uniformity of* $\mathcal{A}$ *to* $\mathcal{B}$ *is at most* $\max(n, 2)$.

Levels of uniformity in randomness

Theorem. (Effective 0-1-law, Kučera) *Let $n \in \omega$, let $\mathcal{V}$ be a Π^0_n-class of positive measure and let X be n-random. Then there is a $k \in \omega$ with $X \upharpoonright [k, \infty) \in \mathcal{V}$.*

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Theorem. *Let $n \in \omega$, let $\mathcal{A}$ be a mass problem and let $n-\mathrm{Random}$ be the class of n-randoms. Assume there exists a Π^0_n-class $\mathcal{V}$ of positive measure such that $\mathcal{A} \leq_{\mathcal{M}} \mathcal{V}$. Then $\mathcal{A} \leq_n n\text{-}\mathrm{Random}$.*

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Theorem. *The uniformity of the non-computably-dominated functions to the 2-random sets is 2.*

More levels of uniformity

Theorem. (Higuchi and Kihara)

$$\mathrm{DNC}_2 \not\leq_1 \mathrm{DNC}_3.$$

Corollary. *The uniformity of* DNC_2 *to* DNC_3 *is* 2.

Comparing to layerwise computability

Fix a universal Martin-Löf test $\mathcal{U}_0, \mathcal{U}_1, \ldots$. Let us say that $\mathcal{A}$ *layerwise reduces to* 1*-randomness* if there is a uniformly computable sequence $e_0, e_1, \ldots$ such that $\Phi_{e_i}(2^\omega \setminus \mathcal{U}_i) \subseteq \mathcal{A}$.

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Proposition. *We do not have that n-DNC_{2^m} reduces layerwise to n-randomness.*

Elementary (in)equivalence

Theorem. *Let* $n, m \in \omega \cup \{\infty\}$ *with* $m < n$ *and* $\{n, m\} \neq \{0, 1\}$. *Then* $\mathcal{M}_n$ *and* $\mathcal{M}_m$ *are not elementarily equivalent.*

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Proof. Easy case: $n = \infty$. Muchnik reducibility is definable in $\mathcal{M}_m$ (Dyment). Since m-reducibility and Muchnik reducibility do not coincide, form the sentence expressing this.

Elementary (in)equivalence

Hard case: $n \in \omega$. We use the following two lemmas.

Lemma. *If* f, g *are* Δ^0_n, *then* $C(\{f\}) \otimes C(\{g\}) \equiv_n C(\{f, g\})$.

Lemma. *Let* $X \oplus Y$ *be* $\max(m, 1)$*-random. Then* $C(\{X\}) \otimes C(\{Y\}) \not\leq_m C(\{X, Y\})$.

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- C is definable in the Medvedev degrees (essentially Dyment).
- The Δ^0_n-degrees are definable in the Turing degrees (Shore and Slaman).

Using this, express that "there are Δ^0_n X and Y such that $C(\{X\}) \otimes C(\{Y\}) \not\leq C(\{X, Y\})$". □

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