Bounded low and high sets

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Reducibilities

Compare relative computational complexity of $A, B \subset \omega$.

If $A \leq B$ and $B \leq A$, we say A, B have the same *degree.*

From weaker to stronger:

- *Turing* $\leq_T$
- *Bounded Turing* $\leq_{bT}$ *(or Weak Truth Table* $\leq_{wtt}$*)*
- *Truth table* $\leq_{tt}$
- *m-reducibility* $\leq_m$
- *1-reducibility* $\leq_1$

Bounded Turing reducibility

Let $A, B \subset \omega$.

Let $B \upharpoonright x = \{n \in B \mid n \leq x\}$.

Definition

A is *bounded Turing (weak truth table) reducible to* B, written $A \leq_{bT} B$, if there are i, j s.t.

- φ_j is total and
- for all x, $\quad A(x) = \Phi_i^{B \upharpoonright \varphi_j(x)}(x) \downarrow$.

where

$\varphi_0, \varphi_1, \varphi_2, \ldots$ an effective enum. of the part. comp. fcns. and
$\Phi_0, \Phi_1, \Phi_2, \ldots$ an effective enum. of the Turing functionals.

Motivation

(Anderson & Csima)
Defined a "bounded jump" A^b tailored to work with $\leq_{bT}$ that is strictly increasing and order preserving on bounded Turing degrees.

They studied

- Jump inversion properties
- Generalized result $A \leq_{bT} \emptyset'$ iff A is ω-c.e.

Goals

1. Compare bounded low/high sets to standard jump counterparts.
2. Discuss analogue of Jump Theorem: $A \leq_T B$ iff $A' \leq_1 B'$. Do we have
$$A \leq_{bT} B \iff A^b \leq_1 B^b?$$

Bounded jump

Definition (Anderson & Csima)

The *bounded jump of a set A* is

$$A^b = \{x \in \omega \mid (\exists i \leq x)[\varphi_i(x)\downarrow \ \wedge \ \Phi_x^{A \upharpoonright \varphi_i(x)}(x)\downarrow]\}$$

Let A^{nb} denote the n-th bounded jump.

Lemma (Anderson & Csima)

For all sets A,

- (Strictly increasing) $A \leq_1 A^b \leq_1 A'$ *but* $A^b \not\leq_{bT} A$.
- (Order preserving) *If* $A \leq_{bT} B$, *then* $A^b \leq_{tt} B^b$.
- $\emptyset^b \equiv_1 \emptyset'$.
- $A^b \equiv_T A \oplus \emptyset'$ *but* $B^b \not\equiv_{bT} B \oplus \emptyset'$ *for most* B.

Low & high sets

Recall that a set A is low (high) w.r.t. a jump operator if its jump encodes the least (most) possible information.

So,

- A is *low* if $A' \leq_T \emptyset'$ and *bounded low* if $A^b \leq_{bT} \emptyset^b \equiv_1 \emptyset'$.
- $A \leq_T \emptyset'$ is *high* if $A' \geq_T \emptyset''$, and
- $A \leq_{bT} \emptyset^b \equiv_1 \emptyset'$ is *bounded high* if $A^b \geq_{bT} \emptyset^{2b}$.

Remark $X \leq_T \emptyset^{(n)} \iff X$ is Δ^0_{n+1}.

Anderson & Csima characterized the sets $X \leq_{bT} \emptyset^{nb}$

Ershov hierarchy

Fix a "nice" computable coding of ordinals $< \omega^\omega$.

Let $\alpha \geq \omega$.

Definition

A set A is *α-c.e.* if there is a partial comp. $\psi : \omega \times \alpha \to \{0, 1\}$ s.t.

- for all $n \in \omega$, there is a $\beta < \alpha$ s.t. $\psi(n, \beta)\downarrow$, and
- $A(n) = \psi(n, \gamma)$ for γ least s.t. $\psi(n, \gamma)\downarrow$.

The following generalizes the classical result

$$A \leq_{bT} \emptyset' \iff A \text{ is } \omega\text{-c.e.}$$

Theorem (Anderson & Csima)

For a set X and $n \geq 2$,

$$X \leq_{bT} \emptyset^{nb} \iff X \text{ is } \omega^n\text{-c.e.} \iff X \leq_1 \emptyset^{nb}.$$

Extreme examples

Goals

1. Compare bounded low/high sets to standard jump counterparts.

Are there "extreme" examples – ones that code high information w.r.t. to one jump and low content w.r.t. the other?

Theorem

There exists a c.e. bounded low set that is high.

Requirements

Recall (Martin) $\emptyset'' \leq_T A'$ iff there is a dominant fcn. $f \leq_T A$, i.e., any comp. g satisfies $(\forall^\infty n)[\, f(n) \geq g(n)\,]$.

If $\bar{A} = \{a_0 < a_1 < a_2 <\}$, then $p_{\bar{A}}(n) = a_n$.

($p_{\bar{A}}$ dominant)

$\mathbf{R}_i$: If φ_i is total, then $(\exists m)(\forall l \geq m)[p_{\bar{A}}(l) \geq \varphi_i(l)]$.

(A^b ω-c.e.) To show A bounded low, enforce a computable bound on times $A^b(x)[s]$ can change.

$$x \in A^b \iff (\exists n \leq x)[\varphi_n(x)\downarrow \wedge \Phi_x^{A \upharpoonright \varphi_n(x)}(x)\downarrow]$$

$\mathbf{P}_x$ *protects uses* $\varphi_n(x)$ *for* $A^b(x)$ *if* $n \leq x$.

Low bounded high

Theorem

There exists a low set $A \leq_{bT} \emptyset'$ that is bounded high.

Construct ω-c.e. set A (so $A \leq_{bT} \emptyset'$) satisfying

$$A' \leq_T \emptyset' \text{ and } A^b \geq_{bT} \emptyset^{2b}.$$

Requirements

(A low)

$\mathbf{N}_e$: $\quad (\exists^\infty s)\left[\Phi_e^A(e)[s]\downarrow\right] \implies \Phi_e^A(e)\downarrow.$

(Code $\emptyset^{2b}$ into A^b)

Construct Ψ and comp. f s.t.

$\mathbf{P}_n$: $\quad \Psi^{A^b \upharpoonright f(n)}(n)\downarrow = \emptyset^{2b}(n).$

Use that $\emptyset^{2b}$ is ω^2-c.e.

Low bounded high

Theorem

There exists a low set $A \leq_{bT} \emptyset'$ that is bounded high.

Construct ω-c.e. set A (so $A \leq_{bT} \emptyset'$) satisfying

$$A' \leq_T \emptyset' \text{ and } A^b \geq_{bT} \emptyset^{2b}.$$

(A ω-c.e.) Make sure $A(n)[s]$ changes at most n times.

Requirements

(A low)

$\mathbf{N}_e$: $\quad (\exists^\infty s)\left[\Phi_e^A(e)[s]\downarrow\right] \implies \Phi_e^A(e)\downarrow.$

$\star$ Restrain $\mathbf{P}_i$ with $i \geq e$ from changing $A(x)[s]$ below

$$r(e,s) = max_{t \leq s} \text{ use of } \Phi_e^A(e)[t].$$

Low bounded high

Theorem
There exists a low set $A \leq_{bT} \emptyset'$ that is bounded high.

(Code $\emptyset^{2b}$ into A^b)

Construct Ψ and comp. f s.t.

$\mathbf{P}_n$: $\Psi^{A^b \upharpoonright f(n)}(n) \downarrow = \emptyset^{2b}(n)$.

$\emptyset^{2b}$ is ω^2-c.e. so $\emptyset^{2b}$ has an approx fcn $\psi(n, \beta)$

- Let $\beta_{n,s}$ be least $\beta < \omega^2$ s.t. $\psi(n, \beta)[s] \downarrow$.
- $\beta_{n,s} = \omega \cdot l_{n,s} + k_{n,s}$ for some $l_{n,s}, k_{n,s} \in \omega$.

Low bounded high

(Code $\emptyset^{2b}$ into A^b)

$\mathbf{P}_n$: $\quad \Psi^{A^b \upharpoonright f(n)}(n) \downarrow = \emptyset^{2b}(n)$.

$\Psi^{A^b}(n)$ only asks about $\mathbf{P}_n$**-block of indices** (which we control).

Block consists of **location** / $\mathbf{P}_n$**-subblocks**, for $l \leq l_{n,0}$, each with

- Location index: indicates if $l_{n,s} = l$ at current stage s;
- Coding indices: If $l_{n,s} = l$, then $c_{n,s}$ codes $\emptyset^{2b}(n)[s]$;
- Injury indices: Use to destroy Ψ-computations if injury.

(Can find comp. f since size of blocks computably bounded.)

Low bounded high

(Code $\emptyset^{2b}$ into A^b) Assume

$$\mathbf{P}_n[s]: \quad \Psi^{A^b}(n)[s]\downarrow = A^b(c_{n,s})[s] = \emptyset^{2b}(n)[s], \text{ and}$$

$$c_{n,s} \in A^b[s] \text{ depends on if } \tilde{c}_{n,s} \in A[s].$$

stage $s+1$ Consider if $l_{n,s+1} = l_{n,s}$:

Yes Continue in Location $l_{n,s}$-subblock & Maintain Ψ.

- If $k_{n,s+1} = k_{n,s}$, maintain status quo.
- Else, change $A(\tilde{c}_{n,s})[s+1]$ to update if coding index $c_{n,s} \in A^b[s]$.

No Kill Ψ-computation with Location index, update it in Location $l_{n,s+1}$-subblock

$\star$ $\mathbf{P}_n$ injured if $\tilde{c}_{n,s} \le \max_{e \le n} r(e, s+1)$

Use Injury indices to change Ψ, $c_{n,s}$ $\tilde{c}_{n,s}$.

Low bounded high

Theorem

There exists a low set $A \leq_{bT} \emptyset'$ that is bounded high.

Theorem

There exists a c.e. bounded low set that is high.

Question

Does there exist a low **c.e.** *set that is bounded high?*

Jump inversion results

Theorem (Schoenfield)
For all Σ_2 sets $X \geq_T \emptyset'$, there is a $Y \leq_T \emptyset'$ s.t. $Y' \equiv_T X$.

Analog of **Schoenfield jump inversion** fails with usual jump.

Theorem (Csima, Downey, & Ng)
There is a Σ_2 set $C >_{tt} \emptyset'$ s.t. for all $D \leq_T \emptyset'$ we have $D' \not\equiv_{bT} C$.

Jump inversion results

Theorem (Schoenfield)

For all Σ_2 sets $X \geq_T \emptyset'$, there is a $Y \leq_T \emptyset'$ s.t. $Y' \equiv_T X$.

Analog of **Schoenfield jump inversion** fails with usual jump.

But...

Analog of **Schoenfield jump inversion** holds with bounded jump.

Theorem (Anderson & Csima)

Given B s.t. $\emptyset^b \leq_{bT} B \leq_{bT} \emptyset^{2b}$,
there is an $A \leq_{bT} \emptyset^b$ s.t. $A^b \equiv_{bT} B$.

What other classical results hold for bounded Turing degrees?

Jump Theorem

Let A, B be sets.

Theorem (Jump Theorem)

$$A \leq_T B \Leftrightarrow A' \leq_1 B'$$

Another bounded jump:

$$A^{b_0} = \{\langle e, i, j\rangle \in \omega \mid \varphi_i(j)\downarrow \ \wedge \Phi_e^{A\restriction\varphi_i(j)}(j)\downarrow\}.$$

Theorem

1. (Anderson & Csima) $A \leq_{bT} B \implies A^{b_0} \leq_1 B^{b_0}$.
2. (Downey & Greenberg) *Converse false: there are A, B s.t. $A^{b_0} \leq_1 B^{b_0}$ but $A \not\leq_{bT} B$.*

Analog for bounded jump does not follow:

(Anderson & Csima) $A^{b_0} \equiv_{tt} A^b$ but $A^b \not\equiv_1 A^{b_0}$ possible.

Bounded Jump Theorem

Let A, B be sets.

Theorem (Jump Theorem)

$$A \leq_T B \Leftrightarrow A' \leq_1 B'$$

Theorem

1. $A \leq_{bT} B \implies A^b \leq_1 B^b$.
2. *Converse false: there are c.e. sets A, B s.t.*

$$A^b \leq_1 B^b \text{ but } A \not\leq_{bT} B.$$

More Questions

Definition

A set $A \leq_T \emptyset'$ is *superlow* if $A' \leq_{tt} \emptyset'$ and *superhigh* if $A' \geq_{tt} \emptyset''$.

(Mohrherr) There are low but not superlow and high but not superhigh c.e. sets.

Question

Does there exist a low bounded low set that is not superlow? Does there exists a high bounded high set that is not superhigh?

Any superlow set is bounded low since $A^b \leq_1 A'$.
But, does there exist a superhigh set that is not bounded high?

Question

Provide characterizations of bounded low and bounded high sets.

Ask these questions for other strong reducibility jump operators, (e.g., Gerla, etc.)

References

Truth table reducibility

Definition

A *truth table reduction* is a pair of computable functions f and g s.t for all x,

- $f(x)$ supplies a finite list $x_1, \dots, x_n$ of oracle positions, and
- $g(x)$ gives a truth table on n variables (a map $2^n \to 2$).

A is *truth table reducible* to B, written $A \leq_{tt} B$, if there is a truth-table reduction f, g s.t. for all x,

$$x \in A \iff$$

the row of the truth table g obtained by viewing B on the positions $x_1, \dots, x_n$ outputs value 1.

Gerla's jump operator

Definition

A *tt-condition* consists of an $(x_1 \dots x_k) \in \omega^{<\omega}$ and an $\alpha : 2^k \to 2$.

A *tt*-condition is *satisfied by A* if $\alpha(A(x_1) \dots A(x_k)) = 1$.

We set $A^{tt} = \{x \mid x \text{ is a } tt\text{-condition satisfied by } A\}$.

Note $A^{tt} \leq_{tt} A$ and $A \leq_1 A^{tt}$.

Definition (Gerla)

Jumps for

- (*tt*-degrees) $A_{tt} = \{x \mid \varphi_x(x)\downarrow \in A^{tt}\}$.
- (bounded *tt*-degrees of norm $k \in \omega$) $A_{bk} = \{x \mid \varphi_x(x)\downarrow \in A^{tt} \ \wedge \ \varphi_x(x) \leq k\}$.

Results on Gerla's jump

Theorem (Gerla)

1. $A_{tt} \not\leq_{tt} A$. $A_{bk} \not\leq_{bk} A$.
2. $A \leq_{tt} B \Rightarrow A_{tt} \leq_1 B_{tt}$.
3. $A <_1 A_{bk} \leq_1 A_{b(k+1)} \leq_1 A_{tt} \leq_1 A'$.
4. $\emptyset_{bk} \equiv_1 \emptyset_{tt} \equiv_1 \emptyset'$.

Theorem (Gerla)

If A is n-c.e. and $B \leq_1 A_{bk}$, then B is $(nk+1)$-c.e.

Theorem (Anderson &Csima)

In general, $A_{tt} \leq_1 A^b$, but there are many X s.t. $X^b \not\leq_{bT} X_{tt}$.