Computable copies of ℓ^p

Timothy H. McNicholl

Department of Mathematics
Iowa State University
mcnichol@iastate.edu

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- Today I will discuss some contributions to a recent program (launched by Melnikov, Nies,Ng) that applies computable structure theory to computable analysis.
- Specifically, I will discuss the application of ‘computable categoricity’ to Banach spaces. (We only consider Banach spaces over $\mathbb{C}$ but all results are still true with $\mathbb{C}$ replaced by $\mathbb{R}$.)
- All non-trivial Banach spaces are uncountable so the usual definitions from computable structure theory require some modification.
- To see how to do this, let’s first discuss how to do computability on Banach spaces. There are two ingredients: generating sets and effective generating sets.

Computability on Banach spaces: generating sets

Suppose $\mathcal{B} = (V, +, \cdot, \| \ \|)$ is a Banach space. Let $S \subseteq V$.

- $\mathcal{L}(S) :=$ the linear span of S. $\overline{\mathcal{L}(S)} =$ the closed linear span of $S =$ the *subspace generated by* S.
- When K is a subfield of $\mathbb{C}$, the *span of* S *over* K is

$$\mathcal{L}_K(S) = \left\{ \sum_{j=0}^{M} \alpha_j v_j \ : \ \alpha_0, \ldots, \alpha_M \in K \ \wedge \ v_0, \ldots, v_M \in S \right\}$$

  (So $\mathcal{L}(S) = \mathcal{L}_{\mathbb{C}}(S)$.)
- S is a *generating set* for $\mathcal{B}$ if V is the closed linear span of S.

An example

Suppose $1 \leq p < \infty$.

1. Recall that ℓ^p is the set of all sequences of complex numbers $\{a_n\}_{n=0}^{\infty}$ so that

$$\sum_{n=0}^{\infty} |a_n|^p < \infty.$$

2. ℓ^p is a Banach space under the norm

$$\|\{a_n\}_n\|_p = \left(\sum_{n=0}^{\infty} |a_n|^p\right)^{1/p}.$$

3. Let $e_n =$ the vector in ℓ^p whose $(n+1)$st component is 1 and whose other components are 0. Let $E = \{e_n \ : \ n \in \mathbb{N}\}$. Then, E is a generating set for ℓ^p. We call E the *standard generating set* for ℓ^p. $E =$ default generating set.

Computability on Banach spaces: effective generating sets

Suppose $\mathcal{B} = (V, +, \cdot, \| \ \|)$ is a Banach space. Let $F = \{f_0, f_1, \ldots\}$ be a generating set for $\mathcal{B}$.

- F is an *effective generating set* if $\| \ \|$ is computable on $\mathcal{L}_{\mathbb{Q}(i)}(F)$. i.e. there is an algorithm that given $k \in \mathbb{N}$ and $f \in \mathcal{L}_{\mathbb{Q}(i)}(F)$ computes $q \in \mathbb{Q}$ so that $|q - \|f\|| < 2^{-k}$.
- Since $\| \ \|$ is computable on $\mathcal{L}_{\mathbb{Q}(i)}(F)$, this gives us a computable metric space. So concepts like 'computable vector (with respect to F)', 'computable operator with respect to (F_1, F_2)' etc. are now defined.
- If F is an effective generating set for $\mathcal{B}$, then $(\mathcal{B}, F)$ is called a *computable Banach space*.

A few examples

Suppose p is a computable real.

- Example: the standard generating set for ℓ^p is an effective generating set. A vector is computable with respect to E if and only if it is a computable sequence of complex numbers.
- Another example: suppose $|\zeta| = 1$, and set $F = \{\zeta e_0, \zeta e_1, \ldots\}$.
  - Then, F is an effective generating set for ℓ^p.
  - In addition, ζe_0 is computable with respect to F. (Even if ζ is incomputable!)
  - The map $T : \ell^p \to \ell^p$ given by $T(f) = \zeta f$ is computable with respect to (E, F).

Isometries (when are two Banach spaces 'identical'?)

Suppose $\mathcal{B}_0$ and $\mathcal{B}_1$ are Banach spaces.

- Recall that a map $T : \mathcal{B}_0 \to \mathcal{B}_1$ is an *isometry* if $\|T(u) - T(v)\|_{\mathcal{B}_1} = \|u - v\|_{\mathcal{B}_0}$ whenever $u, v \in \mathcal{B}_0$.
- Isometries are one-to-one.
- If $T : \mathcal{B}_0 \to \mathcal{B}_1$ is linear, T is an isometry if and only if T preserves the norm. i.e. $\|T(u)\|_{\mathcal{B}_1} = \|u\|_{\mathcal{B}_0}$ whenever $u \in \mathcal{B}_0$.
- $\mathcal{B}_0$ and $\mathcal{B}_1$ are 'identical' if there is a linear isometry of one onto the other. i.e. if they are *linearly isometric*.

Definition (A.G. Melnikov 2013)

Suppose $\mathcal{B}$ is a computable Banach space. $\mathcal{B}$ is *computably categorical* if it is the case that whenever $\mathcal{B}_0$ and $\mathcal{B}_1$ are computable Banach spaces that are linearly isometric to $\mathcal{B}$, there is a computable linear isometry of $\mathcal{B}_0$ onto $\mathcal{B}_1$.

Thus, $\mathcal{B}$ is computably categorical if and only if it is the case that whenever F_0 and F_1 are effective generating sets for $\mathcal{B}$, there is an isometric endomorphism of $\mathcal{B}$ that is computable with respect to (F_1, F_2).

Prior results and questions

- Pour-El and Richards showed that ℓ^1 is not computably categorical. i.e. there is a computable Banach space $\mathcal{B}$ that is linearly isometric to ℓ^1 but not by any computable linear isometry.
- It follows from a recent result of Melnikov that ℓ^2 is computably categorical. Proof sketch: since ℓ^2 is a Hilbert space.
- Question (Melnikov): what about other values of p?

New results

Theorem 1 (M. 2015)

Suppose p is a computable real so that $p \geq 1$ and $p \neq 2$. Suppose C is a c.e. set. Then, there is a computable copy of ℓ^p, $\mathcal{B}$, so that C computes a linear isometry of ℓ^p onto $\mathcal{B}$ and so that any oracle that computes a linear isometry of ℓ^p onto $\mathcal{B}$ must also compute C.

Corollary (M. 2015)

Suppose p is a computable real so that $p \geq 1$. Then, ℓ^p is computably categorical if and only if $p = 2$.

Theorem 2 (M. 2015)

Suppose p is a computable real so that $p \geq 1$, and suppose $\mathcal{B}_0$ and $\mathcal{B}_1$ are computable copies of ℓ^p. Then, the halting set computes a linear isometry of $\mathcal{B}_0$ onto $\mathcal{B}_1$.

The proof of the first theorem is based on the following.

Theorem (Banach/Lamperti)

Suppose p is a real number so that $p \geq 1$ and $p \neq 2$. Let T be a linear map of ℓ^p into ℓ^p. Then, the following are equivalent.

1. *T is a surjective isometry.*
2. *There is a permutation of $\mathbb{N}$, ϕ, and a sequence of unimodular points, $\{\lambda_n\}_n$, so that $T(e_n) = \lambda_n e_{\phi(n)}$ for all n.*

The proof of this theorem is based on the following.

Theorem (Banach/Lamperti)

Suppose p is a real number so that $p \geq 1$ and $p \neq 2$. If $T : \ell^p \to \ell^p$ is a linear isometry, then T preserves the 'disjoint support' relation. i.e. for all $f, g \in \ell^p$

$$\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset \Rightarrow \mathrm{supp}(T(f)) \cap \mathrm{supp}(T(g)) = \emptyset.$$

Lamperti's proof of the latter theorem is based on the following.

Theorem (Lamperti 1959)

Suppose $0 < p < \infty$, *and suppose* $z, w \in \mathbb{C}$.

1. *If* $p < 2$, *then* $|z + w|^p + |z - w|^p \leq 2|z|^p + 2|w|^p$.
2. *If* $p > 2$, *then* $|z + w|^p + |z - w|^p \geq 2|z|^p + 2|w|^p$.
3. *In (1) and (2), equality holds if and only if* $zw = 0$.

For the proof of Theorem 2, we use the following sharpening of Lamperti's inequalities. Let $c_p = |4 - 2\sqrt{2}^p|$.

Theorem 3 (M. 2015)

Suppose $0 < p < \infty$, *and suppose* $z, w \in \mathbb{C}$.

1. *If* $p < 2$, *then* $c_p \min\{|z|^p, |w|^p\} \leq 2|z|^p + 2|w|^p - |z + w|^p - |z - w|^p$.
2. *If* $p > 2$, *then* $c_p \min\{|z|^p, |w|^p\} \leq |z + w|^p + |z - w|^p - 2|z|^p - 2|w|^p$.

Proof sketch of Theorem 1

Suppose p is a computable real so that $p \geq 1$ and $p \neq 2$. The proof uses a trick due to Pour-El and Richards and the Banach/Lamperti classification.

- Let C be a c.e. set. We assume C is infinite, and that $0 \notin C$.
- Let $\gamma = \sum_{k \in C} 2^{-k}$. Thus $0 < \gamma < 1$.
- Let $\{c_n\}_{n \in \mathbb{N}}$ be a one-to-one effective enumeration of C.
- Set:

$$
\begin{aligned}
f_0 &= (1-\gamma)^{1/p} e_0 + \sum_{n=1}^{\infty} 2^{-c_{n-1}/p} e_n \\
f_{n+1} &= e_{n+1} \\
F &= \{f_0, f_1, \ldots\}
\end{aligned}
$$

Proof sketch of Theorem 1

F is an effective generating set:

- First note that $\|f_0\|_p^p = (1-\gamma) + \sum_{n=1}^{\infty} 2^{-c_{n-1}} = 1$.
- When calculating $\| \sum_{j=0}^{M} \alpha_j f_j \|_p^p$, only finitely many of the terms in above sum are distorted.
- If each $\alpha_j \in \mathbb{Q}(i)$ we can calculate the effects of these distortions and so compute $\| \sum_{j=0}^{M} \alpha_j f_j \|_p^p$.
- Since p computable, we can then compute $\| \sum_{j=0}^{M} \alpha_j f_j \|_p$.

Proof sketch of Theorem 1:

Suppose an oracle X computes a surjective linear isometry $T : \ell^p \to \ell^p$ with respect to (E, F).

- By Banach/Lamperti $T(e_{j_0}) = \lambda e_0$ for some j_0 and unimodular λ.
- Since e_{j_0} computable with respect to E, X computes λe_0 with respect to F.
- So from $k \in \mathbb{N}$, X can compute $f \in \mathcal{L}_{\mathbb{Q}(i)}(F)$ so that $\|\lambda e_0 - f\|_p^p < 2^{-k}$.
- If $\|\lambda e_0 - \sum_{j=0}^{M} \alpha_j f_j\|_p^p < 2^{-k}$, then $|\lambda - \alpha_0(1-\gamma)^{1/p}|^p < 2^{-k}$, and so $|1 - |\alpha_0|(1-\gamma)^{1/p}|^p < 2^{-k}$.
- So, since X computes T, X computes γ and so X computes C.

Finally, C computes the identity map with respect to (E, F).

Proof sketch Theorem 2

Suppose p is a computable real so that $p \neq 2$ and $p \geq 1$. Suppose $F = \{f_0, f_1, \ldots, \}$ is an effective generating set for ℓ^p. Using oracle $\emptyset'$ we wish to build an isometric endomorphism T of ℓ^p that is computable with respect to (E, F).

1. Define $T(e_0), T(e_1), \ldots$ so that they are disjointly supported unit vectors.
2. Ensure $f_j \in \overline{\mathcal{L}(T[E])}$ for each j.
3. Define T on ℓ^p via linear extension.
4. $T(e_0)$, $T(e_1)$, $\ldots$ are defined by computable Cauchy sequences. $\emptyset'$ is used to compute moduli of convergence.

Proof sketch Theorem 2

- Problem: with respect to F we can see norms of vectors but we can not see components of vectors. We use Theorem 3 to overcome this obstacle as follows.
- When $z, w \in \mathbb{C}$, let

$$\sigma_p(z, w) = c_p^{-1} ||z + w|^p + |z - w|^p - 2|z|^p - 2|w|^p|.$$

  Therefore $zw = 0$ if and only if $\sigma_p(z, w) = 0$ (by Lamperti's inequalities).
- When $f, g \in \ell^p$ let

$$\sigma_p(f, g) = \sum_{j=0}^{\infty} \sigma_p(f(j), g(j)).$$

  Therefore, $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$ if and only if $\sigma_p(f, g) = 0$.

Proof sketch of Theorem 2

But, even more is true. The following is a consequence of Theorem 3.

Theorem (M. 2015)

Suppose $1 \leq p < \infty$ *and* $p \neq 2$. *Suppose* $f, g \in \ell^p$. *Then, there exist disjointly supported vectors* $f_1, g_1 \in \ell^p$ *so that*

$$\max\{\|f_1 - f\|_p^p, \|g_1 - g\|_p^p\} \leq \sigma_p(f, g).$$

This tells us how far we have to go from (f, g) to find a disjointly supported pair of vectors.

Conclusion

Suppose p is a computable real so that $p \geq 1$.

- We have shown that ℓ^p is computably categorical if and only if $p \neq 2$. The proof rests on the Banach/Lamperti characterization of the isometries of ℓ^p.
- Suppose $p \neq 2$. Theorem 1 also entails that computing surjective linear isometries between computable copies of ℓ^p is at least as hard as computing the halting set.
- This is optimal: the halting set can compute a surjective linear isometry between any two computable copies of ℓ^p.
- Question: If $p \neq 2$ and **c** is a c.e. degree, then there are computable copies of ℓ^p $\mathcal{B}_0$ and $\mathcal{B}_1$ so that **c** is the least powerful oracle that computes a linear isometry of $\mathcal{B}_0$ onto $\mathcal{B}_1$. Are there other degrees with this property?