RT^1_k, SRT^2_ℓ and $\leq_{sc}$ reducibility

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Set up for reverse mathematics

Second order arithmetic: number and set variables, $+$, $\cdot$, $\leq$, $\in$, 0, 1.

Z_2: PA^-+ set induction + comprehension scheme.

Models: $\mathcal{M} = (M, S_M, +_M, \cdots)$ with $S_M \subseteq \mathcal{P}(M)$.

Project: Prove implications (or equivalences) between theorems (or subsystems) over a weak base theory.

RCA_0: $PA^- + \Sigma^0_1$ induction $+\Delta^0_1$ comprehension scheme.

If $M = \omega$, $\mathcal{M}$ is an ω-model and we identify it with $S \subseteq \mathcal{P}(\omega)$.

An ω model satisfies PA^- and full induction.

$(\omega, S) \models RCA_0 \Leftrightarrow S$ closed under $\oplus$ and $\leq_T$.

REC is the minimal ω-model.

Ramsey's theorem for singletons and induction

$\mathsf{RT}^1_k : \forall f : \mathbb{N} \to k \, \exists X \, (X \text{ infinite } \wedge f \restriction X \text{ constant})$

For all $k \in \omega$, $\mathsf{REC} \models \mathsf{RT}^1_k$ and $\mathsf{RCA}_0 \vdash \mathsf{RT}^1_k$.

$\mathsf{RT}^1_{<\infty} : \forall k \, (\mathsf{RT}^1_k)$

Again, $\mathsf{REC} \models \mathsf{RT}^1_{<\infty}$.

Theorem (Hirst)

Over RCA_0, $\mathsf{RT}^1_{<\infty}$ *is equivalent to* $B\Sigma^0_2$.

For any ω-model S of RCA_0, $S \models \mathsf{RT}^1_{<\infty}$.

But $\mathsf{RCA}_0 \not\vdash \mathsf{RT}^1_{<\infty}$.

COH

COH: For every sequence of sets $\langle R_i \mid i \in \mathbb{N} \rangle$, there is an infinite C s.t.

$$\forall i \, (C \subseteq^* R_i \vee C \subseteq^* \overline{R}_i)$$

You can think of COH as an infinite collection of RT^1_2 instances

$$f_i(x) = \begin{cases} 0 & \text{if} \quad x \in \overline{R}_i \\ 1 & \text{if} \quad x \in R_i \end{cases}$$

to solve simultaneously but each allowing finitely many errors. Or as an infinite collection of RT^1_4 instances

$$f_i(x) = \begin{cases} 0 & \text{if} \quad x \in \overline{R}_{2i} \cap \overline{R}_{2i+1} \\ 1 & \text{if} \quad x \in \overline{R}_{2i} \cap R_{2i+1} \\ 2 & \text{if} \quad x \in R_{2i} \cap \overline{R}_{2i+1} \\ 3 & \text{if} \quad x \in R_{2i} \cap R_{2i+1} \end{cases}$$

RT^2_2, SRT^2_2 and D^2_2

Let $f : [\mathbb{N}]^2 \to 2$. H is homogeneous for f if $f \upharpoonright [H]^2$ is constant.

$$\mathsf{RT}^2_2 : \quad \forall f : [\mathbb{N}]^2 \to 2 \; \exists H \, (H \text{ is infinite and homogenous})$$

(RT^2_2 can be proved using $\omega + 1$ successive applications of RT^1_2.)

We say f is stable if $\lim_y f(x,y)$ exists for every x.

$$\mathsf{SRT}^2_2 : \quad \forall \text{ stable } f : [\mathbb{N}]^2 \to 2 \; \exists H \, (H \text{ is infinite and homogenous})$$

We say H is limit-homogeneous for a stable f if there is a color i such that $\lim_y f(x,y) = i$ for every $x \in H$.

$$\mathsf{D}^2_2 : \quad \forall \text{ stable } f : [\mathbb{N}]^2 \to 2 \; \exists H \, (H \text{ is infinite and limit homogenous})$$

Connecting D^2_2, RT^2_2, SRT^2_2 and COH

D^2_2 is equivalent to SRT^2_2 over $RCA_0 + B\Sigma^0_2$.

(Cholak, Jockusch and Slaman) RT^2_2 is equivalent to $SRT^2_2 + COH$ over RCA_0. COH does not imply RT^2_2 over RCA_0 (for induction reasons).

(Hirschfeldt and Shore) There is an ω-model of $RCA_0 + COH$ which does not satisfy RT^2_2.

(Chong, Slaman and Yang) There is a nonstandard model of $RCA_0 + SRT^2_2$ which is not a model of RT^2_2.

Question: Does SRT^2_2 imply RT^2_2 on ω-models of RCA_0?

ω-reducibility

Let P be a true Π^1_2 sentence of the form

$$\forall X \left(\varphi(X) \rightarrow \exists \widehat{X}\, \psi(X, \widehat{X})\right)$$

where φ and ψ are arithmetic statements. We refer to an X such that $\varphi(X)$ as a P-instance and we refer to the corresponding witnesses $\widehat{X}$ such that $\psi(X, \widehat{X})$ as solutions to X.

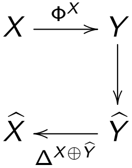
Given two principles of this form P and Q, rather than asking if $\mathsf{RCA}_0 \vdash Q \rightarrow P$, we ask how they compare on ω models.

$$P \leq_\omega Q \Leftrightarrow \forall \text{ Turing ideal } S \left(S \models Q \rightarrow S \models P\right)$$
$$\Leftrightarrow \text{ every } \omega\text{-model of } \mathsf{RCA}_0 + Q \text{ is an } \omega\text{-model of } \mathsf{RCA}_0 + P$$

Question: Does $\mathsf{RT}^2_2 \leq_\omega \mathsf{SRT}^2_2$? Does $\mathsf{COH} \leq_\omega \mathsf{SRT}^2_2$?

Stronger reducibility: $P \leq_c Q$

$P \leq_c Q \Leftrightarrow$ for every P-instance X, there is a Q-instance $Y \leq_T X$ such that for every solution $\widehat{Y}$ of Y, there is a solution $\widehat{X}$ of X with $\widehat{X} \leq_T X \oplus \widehat{Y}$.



Example 1

$\mathsf{SRT}^2_2 \leq_c \mathsf{D}^2_2$

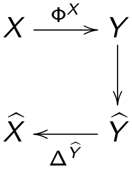
Given an SRT^2_2 instance $f : [\omega]^2 \to 2$, we view f as a D^2_2 instance. Let H be an infinite limit-homogeneous set for f with color i. We thin H to a homogeneous set:

- Set $h_0 =$ the least element of H.
- Let $h_{n+1} =$ the least element $x \in H$ such that $x > h_n$ and $f(h_m, x) = i$ for all $m \leq n$.
- The set $\{h_0, h_1, \ldots\}$ is homogeneous for f.

This procedure uses both the D^2_2 solution H and the SRT^2_2 instance f to compute the SRT^2_2 solution to f.

Stronger reducibility: $P \leq_{sc} Q$

$P \leq_{sc} Q \Leftrightarrow$ for every P-instance X, there is a Q-instance $Y \leq_T X$ such that for every solution $\widehat{Y}$ of Y, there is a solution $\widehat{X}$ of X with $\widehat{X} \leq_T \widehat{Y}$.



Example 2

$\mathsf{RT}^1_{<\infty} \leq_{sc} \mathsf{RT}^2_2$

Given $f : \omega \to k$. Define f-computable coloring $g : [\omega]^2 \to 2$

$$g(x,y) = \begin{cases} 0 & \text{if} \quad f(x) = f(y) \\ 1 & \text{if} \quad f(x) \neq f(y) \end{cases}$$

Let H be an infinite homogeneous set for H. We must have $g \upharpoonright [H]^2 = 0$, so H is homogeneous for f as well.

Once we have RT^2_2 solution H, we do not need to use $\mathsf{RT}^1_{<\infty}$ instance f to help compute $\mathsf{RT}^1_{<\infty}$ solution.

Example 3

$\mathsf{RT}^1_k \leq_{sc} \mathsf{D}^2_k \leq_{sc} \mathsf{SRT}^2_k$

Given $f : \omega \to k$. Define f-computable coloring $g : [\omega]^2 \to k$ by $g(x, y) = f(x)$. The coloring g is stable and any limit-homogeneous set for g is monochromatic for f.

Theorem (Dzhafarov)

- $\mathsf{SRT}^2_2 \not\leq_{sc} \mathsf{D}^2_2$ *and in fact* $\mathsf{SRT}^2_2 \not\leq_{sc} \mathsf{D}^2_{<\infty}$
- $\mathsf{COH} \not\leq_{sc} \mathsf{SRT}^2_2$

A list of questions from Hirschfeldt, Jockusch and Dzhafarov:

1. Motivating Question: Does $\mathsf{RT}^2_2 \leq_\omega \mathsf{SRT}^2_2$? Does $\mathsf{COH} \leq_\omega \mathsf{SRT}^2_2$?
2. Does $\mathsf{COH} \leq_c \mathsf{SRT}^2_2$?
3. Does $\mathsf{RT}^1_3 \leq_{sc} \mathsf{SRT}^2_2$?
4. Does $\mathsf{RT}^1_k \leq_{sc} \mathsf{SRT}^2_\ell$ when $k < \ell$?
5. Does $\mathsf{COH} \leq_{sc} \mathsf{SRT}^2_\ell$ for $\ell > 2$? Does $\mathsf{COH} \leq_{sc} \mathsf{SRT}^2_{<\infty}$?

Theorem (Dzhafarov, Patey, Solomon and Westrick)

- *If $k > \ell$, then* $\mathsf{RT}^1_k \not\leq_{sc} \mathsf{SRT}^2_\ell$.
- $\mathsf{COH} \not\leq_{sc} \mathsf{SRT}^2_{<\infty}$ *(almost certainly).*

Thank you!