# Defining the jump classes in the local structure of the enumeration degrees

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# A quick reminder

## Definition

$A \leq_e B$ if there is a c.e. set $W$, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \;\&\; D \subseteq B)\}\,.$$

- The enumeration degree of a set $A$ is $d_e(A) = \{B \mid A \leq_e B \;\&\; B \leq_e A\}$.
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- The least element: $\mathbf{0_e}$ is the set of all c.e. sets.
- The least upper bound: $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- The enumeration jump: $d_e(A)' = d_e(K_A \oplus \overline{K_A})$, where $K_A = \{\langle e, x\rangle \mid x \in W_e(A)\}$.

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation.

$$(\mathcal{D}_T, \leq_T, \vee, ', \mathbf{0}_T) \cong (\mathcal{TOT}, \leq_e, \vee, ', \mathbf{0}_e) \subseteq (\mathcal{D}_e, \leq_e, \vee, ', \mathbf{0}_e)$$

# The importance of definability

**Theorem (Slaman, Woodin)**

The following are equivalent for $\mathcal{D} \in \{\mathcal{D}_T, \mathcal{D}_e\}$:

1. $\mathcal{D}$ is rigid, i.e. has no nontrivial automorphisms.
2. The definable relations in $\mathcal{D}$ are the ones induced by degree-invariant definable relations on sets in Second Order Arithmetic.
3. $\mathcal{D}$ is biinterpretable with Second Order Arithmetic.

# Definability in the enumeration degrees

**Theorem (Kalimullin)**

The enumeration jump is definable in $\mathcal{D}_e$.

**Theorem (Cai, Ganchev, Lempp, Miller, S)**

The total enumeration degrees are first order definable in $\mathcal{D}_e$.

**Theorem (Cai, Ganchev, Lempp, Miller, S)**

The image of the relation "c.e. in" under the standard embedding of $\mathcal{D}_e$ in $\mathcal{D}_T$ is first order definable in $\mathcal{D}_e$.

# $\mathcal{K}$-pairs

## Definition

A pair of sets $A$ and $B$ are a $\mathcal{K}$-pair if there is a c.e. set $W$, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

For example if $L_A = \{\sigma \in 2^{<\omega} \mid \sigma \leq_{lex} A\}$ and $R_A = \overline{L_A}$ then

1. $L_A \leq_e A$ (and $R_A \leq_e \overline{A}$);
2. $L_A \oplus R_A \equiv_e A \oplus \overline{A}$;
3. $L_A$ and $R_A$ are a $K$-pair via $W = \{\langle\sigma, \tau\rangle \mid \sigma \leq_{lex} \tau\}$.

## Theorem (Kalimullin)

$A$ and $B$ are a $\mathcal{K}$-pair if and only if $d_e(A) = \mathbf{a}$ and $d_e(B) = \mathbf{b}$ satisfy:

$$\forall \mathbf{x}(\mathbf{x} = (\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x})).$$

# Local and global structural interaction

**Theorem (Slaman, S)**

The structure of the enumeration degrees is rigid if any of the following structures are:

1. The structure $\mathcal{R}$ of the c.e. Turing degrees.
2. The structure $\mathcal{D}_T(\leq \mathbf{0}')$of the $\Delta^0_2$ Turing degrees.
3. The structure $\mathcal{D}_e(\leq \mathbf{0}'_e)$ of the $\Sigma^0_2$ enumeration degrees.

# Definability in $\mathcal{D}_e(\leq \mathbf{0}_e')$

**Theorem (Ganchev, S)**

$\mathcal{K}$-pairs are first order definable in $\mathcal{D}_e(\leq \mathbf{0}_e')$ ...

**Theorem (Cai, Lempp, Miller, S)**

...by the same first order formula as in $\mathcal{D}_e$.

**Theorem (Ganchev, S)**

1. The total enumeration degrees are definable in $\mathcal{D}_e(\leq \mathbf{0}_e')$.
2. The $\text{Low}_1$ enumeration degrees are definable in $\mathcal{D}_e(\leq \mathbf{0}_e')$.

**Corollary (Shore)**

The $\text{Low}_{n+1}$ and $\text{High}_n$ **total** enumeration degrees are first order definable in $\mathcal{D}_e(\leq \mathbf{0}_e')$.

# Total degrees in the local structure

**Theorem (Selman)**

$\mathbf{a} \leq \mathbf{b}$ if and only if every total e-degree above $\mathbf{b}$ is above $\mathbf{a}$.

To what extent do the total enumeration degrees determine $\mathcal{D}_e(\leq \mathbf{0}'_e)$?

Selman's theorem is not true locally: there are degrees $\mathbf{a} < \mathbf{0}'_e$ such that the only total enumeration degree in $\mathcal{D}_e(\leq \mathbf{0}'_e)$ above them is $\mathbf{0}'_e$.

**Theorem (Soskov)**

For every $\mathbf{a} \in \mathcal{D}_e$ there is a total $\mathbf{f} \geq \mathbf{a}$ such that $\mathbf{a}' = \mathbf{f}'$.

Is Soskov's jump inversion theorem true locally?

If it is true, then the jump classes would be fully definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$:

$$\mathbf{a} \in \text{Low}_{n+1} \text{ if and only if there is a total } \mathbf{f} \geq \mathbf{a}\text{, such that } \mathbf{f} \in \text{Low}_{n+1}$$

$$\mathbf{a} \in \text{High}_n \text{ if and only if every total } \mathbf{f} \geq \mathbf{a} \text{ are in High}_n$$

# Getting around the difficult question

### Theorem (Ganchev, S)

For every nonzero degree $\mathbf{a} \leq \mathbf{0}_e'$ bounds a nonzero degree for which Soskov's jump inversion holds locally.
For every nonzero degree $\mathbf{a} \leq_e \mathbf{0}_e'$ there is a nonzero degree $\mathbf{b} \leq \mathbf{a}$ and a total $\mathbf{f}$, such that:

1. $\mathbf{b} \leq \mathbf{f}$.
2. $\mathbf{b}' = \mathbf{f}'$.

### Theorem (Ganchev, Sorbi)

Every nonzero enumeration degree $\mathbf{a}$ bounds a nontrivial initial segment of enumeration degrees whose nonzero elements have all the same jump as $\mathbf{a}$.

# Defining the jump classes in $\mathcal{D}_e(\leq \mathbf{0}_e')$

**Theorem (Ganchev, S)**

All jump classes are first order definable in $\mathcal{D}_e(\leq \mathbf{0}_e')$.

$\mathbf{a} \in \mathrm{Low}_{n+1}$ if and only if

for every nonzero $\mathbf{b} \leq \mathbf{a}$

there is a nonzero $\mathbf{x} \leq \mathbf{b}$ and a total $\mathbf{f} \geq \mathbf{x}$, such that $\mathbf{f} \in \mathrm{Low}_{n+1}$.

$\mathbf{a} \in \mathrm{High}_n$ if and only if

there is a nonzero $\mathbf{b} \leq \mathbf{a}$ such that

for every nonzero $\mathbf{x} \leq \mathbf{b}$ all total $\mathbf{f} \geq \mathbf{x}$ are in $\mathrm{High}_n$.

# Proof flavor

**Theorem (Ganchev, S)**

For every nonzero degree $\mathbf{a} \leq \mathbf{0}'_e$ bounds a nonzero degree for which Soskov's jump inversion hold locally.

*Proof flavor:*
Infinite injury construction, borrowing ideas from 'Sacks Jump inversion', 'Good approximations of $\Sigma^0_2$ sets' , 'The ability to "dump" elements in a constructed set $B = \Gamma(A)$ when they are not needed anymore'.

**Theorem (Ganchev, Sorbi)**

Every nonzero enumeration degree $\mathbf{a}$ bounds a nontrivial initial segment of enumeration degrees whose nonzero elements have all the same jump as $\mathbf{a}$.

*Proof flavor:* $\mathcal{K}$-pairs.

# Jumps of $\mathcal{K}$-pairs

## Lemma (Kalimullin)

If $A$ and $B$ are a $\mathcal{K}$-pair and neither $A$ or $B$ are c.e. then:

1. If $C \leq_e A$ then $C$ and $B$ are a $\mathcal{K}$-pair.
2. $A \leq_e \overline{B} \leq_e A \oplus \emptyset'$.

If $A$ and $B$ are a nontrivial $\mathcal{K}$-pair then

$$A' = K_A \oplus \overline{K_A} \leq_e A \oplus (B \oplus \emptyset') \leq_e A \oplus \overline{A} \oplus \emptyset' \leq_e A'.$$

So $A' \equiv_e A \oplus B \oplus \emptyset'$. Similarly $B' \equiv_e A \oplus B \oplus \emptyset'$.

If $C \leq_e A$ and $C$ is not c.e. then $C' \equiv_e B' \equiv_e A'$.

Finally consider an arbitrary non-low set $A$ and the corresponding set $L_{K_A}$.

1. $L_{K_A} \leq_e K_A \equiv_e A$;
2. $L'_{K_A} \equiv_e L_{K_A} \oplus R_{K_A} \oplus \emptyset' \equiv_e K_A \oplus \overline{K_A} = A'$.

# The end

Thank you!