Scott Sentences and Torsion Groups

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What's a Scott Sentence?

Everything in this talk is motivated by a theorem of Dana Scott's:

Scott's Isomorphism Theorem

Every countable structure A can be described up to isomorphism (among countable structures) by a sentence φ of $L_{\omega_1\omega}$.

Such a sentence is called a **Scott sentence** for A.

Complexity of formulas in $L_{\omega_1\omega}$

Every formula of $L_{\omega_1\omega}$ has a normal form.

- A $\Sigma_0 = \Pi_0$ formula is a finitary quantifier-free formula of L.
- A Σ_α formula is a formula of the form $\bigvee_{i\in\omega} \exists \bar{x}_i \phi_i(\bar{x}_i)$ where each ϕ_i is Π_β for $\beta < \alpha$.
- A Π_α formula is the negation of a Σ_α formula. Equivalently, a formula of the form $\bigwedge_{i\in\omega} \forall \bar{x}_i \phi_i(\bar{x}_i)$ where each ϕ_i is Σ_β for $\beta < \alpha$.

A d-Σ_α formula is a finite conjunction of Π_α and Σ_α formulas

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Disagreement in the Literature

Unfortunately, many non-equivalent definitions of Scott rank exist in the literature. Antonio Montalbán in "A Robuster Scott Rank" argued to standardize the following definition:

Definition (A. Montalbán)

The **(categoricity) Scott rank** of A is the least α such that A has a $\Pi_{\alpha+1}$ Scott sentence.

The List

Montalbán believed this notion was most robust, having many other conditions equivalent to it.

Theorem

For any structure A, the following are equivalent:

1. A has a $\Pi_{\alpha+1}$ Scott sentence.
2. The automorphism orbit of any tuple can be defined by a Σ_α formula (without parameters).
3. The set $Iso(A)$ of presentations of A is $\Pi_{\alpha+1}$ in the Borel hierarchy.
4. A is uniformly boldface Δ_α-categorical.
5. And so on...

In other words, Scott sentences are related to notions in computability theory and descriptive set theory.

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Due to a result of A. Miller, there is a unique least-complexity Scott sentence for the structure ($\Pi_\alpha, \Sigma_\alpha, d$-$\Sigma_\alpha$).

Definition (R.A*, M. Harrison-Trainor, D. Turetsky, N. Greenberg)

The **Scott complexity** of a structure A is the least complexity of a Scott sentence for A.

Scott complexity is finer than Scott rank, and just as robust.

Torsion Abelian Groups: The Plan

Wherever in mathematics there is a characterization theorem of a class of structures, there is typically a way to compute the Scott complexity for structures in that class. The torsion abelian groups admit an interesting classification.

Every abelian group G can be written uniquely as $G = D \oplus R$ where D is divisible and R is reduced (no divisible subgroups.) Every reduced torsion group can be further decomposed uniquely as a direct sum of primary groups. There is a structure theorem for the divisible abelian groups. Ulm's theorem states that every countable reduced abelian p-group is determined by its Ulm invariants.

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Torsion Abelian Groups: The Plan

To get at the Scott complexity of an arbitrary torsion abelian group, we can do the following.

1. Compute the Scott complexity of an arbitrary divisible abelian group.
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Reduced Abelian p-Groups

A p-**group** is one in which every element has order p^n for some n.

For a p-group G and an ordinal α, define G_α to be the subgroup of G of elements of height at least α. Define P_α to be the subgroup of G_α of elements of order p. The dimension of $P_\alpha/P_{\alpha+1}$ (as a vector space over $\mathbb{Z}_p$) is called the αth **Ulm invariant** of G.

Theorem

(Ulm's Theorem) Two reduced countable p-groups are isomorphic iff they have the same Ulm invariants.

Note that the G_α's form a descending chain. The stage at which this chain terminates in the trivial group is the **length** of the group.

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Using Ulm's theorem, we can give an upper bound on the Scott complexity of a reduced abelian p-group.

Theorem (R.A.*, *B.Csima*, *L.MacLean*)

Let G be a group with length $\lambda > 0$ and Ulm invariants given by f. Then for $0 < n < \omega$,

(i) If $\lambda = \omega \cdot \gamma + n$ and $f(\omega \cdot \gamma + k) < \infty$ for $0 \leq k < n$ then G has a Scott sentence of complexity d-$\Sigma_{2\gamma+2}$.

(ii) If $\lambda = \omega \cdot \gamma + n$ and $f(\omega \cdot \gamma + i) = \infty$ for some i with $0 \leq i < n$ then G has a Scott sentence of complexity $\Pi_{2\gamma+3}$.

(iii) If $\lambda = \omega \cdot \gamma$ then G has a Scott sentence of complexity $\Pi_{2\gamma+1}$.

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This upper bound on the Scott complexity is in general the best we can do.

Example (R.A.*, *B.Csima*, *L.MacLean*)

Suppose that G is a reduced abelian p-group of finite length with at least two infinite nonzero Ulm invariants. Then G has Scott complexity Π_3.

To prove this, we show that G is the union of a a certain elementary chain.

We will see later a much stronger version of this result, which shows that our upper bound on the Scott complexity is optimal for structures of arbitrarily high length and Scott rank.

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However, there are also instances where this upper bound is not optimal. This happens when there are "nicer" ways to determine the isomorphism type of G than the Ulm invariants.

Example (R.A.*, *B.Csima*, *L.MacLean*)

The group $G = (\mathbb{Z}_{p^n})^\omega$ has Scott complexity Π_2. (Our upper bound only shows the Scott complexity is at most Π_3.)

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Reduced Abelian p-Groups

To show the Scott sentences we obtained are often optimal, we gave a characterization of the back-and-forth relations on reduced abelian p-Groups. Recall $A \leq_\alpha B$ if every Π_α sentence true of A is true of B iff every Σ_α sentence true of B is true of A.

For example, to show that a structure A with a $\Pi_{\alpha+1}$ Scott sentence has a $\Pi_{\alpha+1}$ Scott complexity, it is enough to find a nonisomorphic structure B with $A \equiv_\alpha B$. To show that a structure A with a d-Σ_α Scott sentence has exactly that Scott complexity, it is enough to find a nonisomorphic structures B, C with $B \leq_\alpha A \leq_\alpha C$.

The desired structures are easy to find when we have an algebraic characterization of when the back and forth relations hold.

Characterizing the back-and-forth relations on a class of structures is also of interest in its own right with many applications.

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Theorem (R.A.*, *B.Csima*, *L.MacLean*)

Let A, B be reduced Abelian p-groups of lengths λ_A, λ_B respectively, whose Ulm invariants are given by f_A, f_B.

1. $A \leq_{2\alpha+1} B$ if and only if
   a) for all $\beta < \omega \cdot \alpha$ we have $f_A(\beta) = f_B(\beta)$,
   b) $\lambda_A = \lambda_B < \omega \cdot \alpha$, or $\lambda_B \geq \omega \cdot \alpha$ and $\lambda_A \geq \min\{\lambda, \omega \cdot \alpha + \omega\}$,
   c) $|P^A_\beta| = \infty$ for all $\beta < min\{\lambda_A, \omega\alpha + \omega\}$, or $A_{\omega\alpha} \leq_1 B_{\omega\alpha}$
2. $A \leq_{2\alpha+2} B$ if and only if
   a) for all $\beta < \omega \cdot \alpha$ we have $f_A(\beta) = f_B(\beta)$
   and for $\omega \cdot \alpha \leq \beta < \omega \cdot \alpha + \omega$ we have $f_B(\beta) \leq f_A(\beta)$,
   b) $\lambda_A = \lambda_B < \omega \cdot \alpha + \omega$ or $\omega \cdot \alpha + \omega \leq \lambda_A, \lambda_B$,
   c) $|P^A_\beta| = \infty$ for all $\beta < min\{\lambda, \omega\alpha + \omega\}$, or $B_{\omega\alpha} \leq_2 A_{\omega\alpha}$
3. $A \leq_\alpha B$ for α a limit ordinal or zero if and only if
   a) for all $\beta < \omega \cdot \alpha$ we have $f(\beta) = f(\beta)$,
   b) $\lambda = \lambda < \omega \cdot \alpha$ or $\omega \cdot \alpha \leq \lambda, \lambda$

Reduced Abelian p-Groups

Corollary

Suppose that A is a reduced abelian p-group with $\lambda_A = \omega \cdot \alpha$ where α is a limit ordinal. Then A has Scott complexity $\Pi_{2\alpha+1}$.

For every limit ordinal, there is a reduced abelian p-group with that length. Thus there are reduced abelian p-groups of arbitrarily high Scott complexity.

Thanks

Thank You!