Elementarity of Subgroups of Profinite Permutation Groups via Tree Presentations

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Profinite Groups

A topological group is called profinite if it is isomorphic to the inverse limit of an inverse system of discrete finite groups.

Examples:

- Finite groups
- Direct products of finite groups
- The p-adic integers $\mathbb{Z}_p$ under addition
- Absolute Galois groups

In [2], R. Miller investigated the absolute Galois group of $\mathbb{Q}$ (that is, $\text{Aut}\left(\overline{\mathbb{Q}}\right)$) viewed as a subgroup of S_ω (the group of permutations of $\mathbb{N}$).

Approach

- Although S_ω is size continuum, both it and its closed subgroups can be presented as the set of paths through a countable tree.
- The subgroups of S_ω that can be presented this way with finite branching trees are exactly the profinite ones.
- We can use these presentations to find the complexities of the theories of profinite G as well as see to what degree certain countable subgroups of G will be elementary subgroups.

Tree Presentations

Definition

Let G be a subgroup of S_ω. We define the tree T_G to be the subtree of $\mathbb{N}^{<\omega}$ containing all initial segments of elements of G. That is,

$$T_G := \{\tau \in \mathbb{N}^{<\omega} : (\exists g \in G, n \in \mathbb{N})[\tau = g(0)g(1)\cdots g(n)]\}$$

where $m \in \mathbb{N}$ is mapped to $g(m)$ under g. We define the ordering of T_G via initial segments and write $\tau \sqsubset \sigma$ if τ is an initial segment of σ.

Let G be a subgroup of S_ω. We define the degree of T_G ($\deg(T_G)$) to be the join of the Turing degrees of

- The domain of T_G under some computable coding of $\mathbb{N}^{<\omega}$ in which $\sqsubset$ is decidable; and
- A branching function $Br : T_G \to \mathbb{N} \cup \{\infty\}$ such that $Br(\tau)$ is equal to the number of direct successors of τ in T_G.

Topology

Given a tree $T \subset \mathbb{N}^{<\omega}$, we define $[T]$ to be the set of all paths through T. We endow $[T]$ with the standard product topology in which the basic clopen sets are those of the form $\{f \in \mathbb{N}^{\omega} : \tau \sqsubset f\}$ for some $\tau \in T$.

In order for every path in $[T]$ to represent an element of G, we must have that G is a closed subgroup of S_{ω}.

Profinite Groups and Orbits

Given a subgroup G of S_ω and $n \in \mathbb{N}$, we define the orbit of n under G as

$$\text{orb}_G(n) := \{g(n) \in \mathbb{N} : g \in G\}.$$

Proposition

Let G be a subgroup of S_ω. The following are equivalent:

(1) *G is compact,*

(2) *G is closed and all orbits under G are finite,*

(3) *G is profinite.* □

Orbit Independence

Let G be a profinite subgroup of S_ω. Let $\{O_{G,n}\}_{n\in\mathbb{N}}$ be an enumeration of the orbits under G (all of which are finite). Define

$$H_n := \{g \upharpoonright O_{G,n} : g \in G\}.$$

Definition

We say that G has orbit independence if it is isomorphic to the Cartesian product of all H_n. That is,

$$G \cong \prod_{n\in\mathbb{N}} H_n.$$

A non-example: $G = \{1_G, (0\,1)(2\,3)\}$ does not have orbit independence. Note that $G \cong C_2$, $H_0 = \{1, (0\,1)\} \cong C_2$, $H_1 = \{1, (2\,3)\} \cong C_2$, and H_n is trivial for all $n > 1$. Thus $G \ncong \prod_n H_n \cong C_2 \times C_2$.

Finite Approximations

Let G be a profinite subgroup of S_ω. Given $g \in G$, define $g_k = g \restriction \bigcup_{n \leq k} O_{G,n}$. Define

$$G_k := \{g_k : g \in G\}.$$

Lemma

If G has orbit independence, then given any first order sentence α in the language of groups, $G \models \alpha$ if and only if $G_k \models \alpha$ for all but finitely many k.

This follows from the Feferman-Vaught Theorem.

Complexity of Theories: W/ Orbit Independence

Theorem

Let G be a profinite subgroup of S_ω with orbit independence. The first order theory of G is Δ^0_2 relative to $\deg(T_G)$.

Proof: $G \models \alpha$ iff $(\exists n)(\forall k > n)[G_k \models \alpha]$. Additionally, $G \not\models \alpha$ iff $(\exists n)(\forall k > n)[G_k \models \neg\alpha]$. Thus, both $Th(G)$ and its complement are Σ^0_2 relative to $\deg(T_G)$.

Complexity of Theories: Without Orbit Independence

Proposition

There exist profinite subgroups G of S_ω (without orbit independence) such that the existential theory of G is Σ^0_2-complete relative to $\deg(T_G)$.

This is the worst case scenario for existential theories.

Theorem

Let G be a profinite subgroup of S_ω (not necessarily with orbit independence). The existential theory of G is Σ^0_2 relative to $\deg(T_G)$.

Open Question: How complicated can $Th(G)$ be when G does not have orbit independence?

Turing Ideals and G_I

A collection I of Turing degrees is called a *Turing ideal* if

- I is downwards closed (under $\leq_T$); and
- Given $\boldsymbol{c}, \boldsymbol{d} \in I$, we have $\boldsymbol{c} \oplus \boldsymbol{d} \in I$.

We call I a *Scott ideal* if for every $\boldsymbol{c} \in I$ there exists $\boldsymbol{d} \in I$ that is PA relative to $\boldsymbol{c}$.

Given a subgroup G of S_ω and a Turing ideal I with $\deg(T_G) \in I$, we define G_I to be the subgroup of G all of whose elements are of degree in I. That is,

$$G_I = \{g \in G : \deg(g) \in I\}.$$

Question: To what degree is G_I an elementary substructure of G?

Elementarity

Let $\mathcal{A}$ be a substructure of $\mathcal{B}$ and let Γ be a class of formulas. We say that $\mathcal{A}$ is a Γ-elementary substructure if for all formulas $\gamma \in \Gamma$ and tuples $\bar{a} \in \mathcal{A}$,

$$\mathcal{A} \models \gamma(\bar{a}) \iff \mathcal{B} \models \gamma(\bar{a}).$$

We express this as

$$\mathcal{A} \preceq_{\Gamma} \mathcal{B}.$$

If this holds for all first order formulas γ, then we simply say that $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$ and write

$$\mathcal{A} \preceq \mathcal{B}.$$

Some Results

Proposition

There exists a profinite subgroup G of S_ω such that $G_{\{\mathbf{0}\}}$ is not a $\exists$-elementary subgroup of G.

To prove this, we build a G along with a computable $g \in G$ such that g has a square root in G but no computable square root. This group G will not have orbit independence.

Theorem

Given a profinite subgroup of G with orbit independence and any Turing ideal I,

$$G_I \preceq_\exists G.$$

With Scott ideals

Theorem

Given any profinite subgroup G of S_ω and a Scott ideal I,

$$G_I \preceq_\exists G.$$

Furthermore if G has orbit independence, then

$$G_I \preceq G.$$

Thus, to get that G_I is an elementary subgroup of G it is sufficient to have that I is a Scott ideal and that G has orbit independence.

Open Question: If G has orbit independence but I is not a Scott ideal, or if G does not have orbit independence but I is a Scott ideal, then $G_I \preceq_\exists G$. However, must we have $G_I \preceq G$?

References

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