Some computability theoretic aspects of Dobrinen's theorem that the universal triangle free graph has finite big Ramsey degrees

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https://www.nd.edu/~cholak/amswi24.pdf

Triangle-free Hanson Graph, $\mathcal{H}$

Work with a countable (but not finite) graph $G = (V, E)$.

- 3 nodes form a *triangle* iff they are pairwise connected by edges.
- A graph is *triangle free* iff it contains no triangles.
- A triangle free graph has the *extension property* (or is *homogenous*) iff, for all finite disjoint $D, F \subseteq V$ with no edges among the nodes of D (i.e. D is an anticlique), there is a node n which is connected to all nodes in D (i.e. for all $v \in D$, $(v, n) \in E$) and disconnected from all nodes in F (i.e. for all $v \in F$, $(v, n) \notin E$). We will label this as requirement $\mathcal{R}_{D,F}$.
- Triangle-free Hanson Graph $\mathcal{H}$ is **the** triangle-free homogenous graph (a standard back and forth argument).

Fixing our copy of $\mathcal{H}$

- We will stage-wise construct a graph

$$V = \{\tilde{v}_0, \tilde{v}_1, \dots \tilde{v}_i, \tilde{v}_{i+1}, \dots\},$$

  where we meet all requirements $\mathcal{R}_{D,F}$. Moreover every node in our graph will witness some requirement.
- Now we want to fix an enumeration,

$$V = \{v_0, v_1, \dots v_i, v_{i+1}, \dots\},$$

  of this graph, such that, for all i, v_i and v_{i+1} are connected. (Assume we have v_s. Let k be the first k such $\tilde{v}_k$ is not listed among $\{v_0, v_1, \dots v_s\}$. If $\tilde{v}_k$ and v_i are connected, let $v_{s+1} = \tilde{v}_k$. Otherwise let $\tilde{v}_l$ be such that the pairs v_i and $\tilde{v}_l$ and $\tilde{v}_l$ and $\tilde{v}_k$ are connected and let $v_{s+1} = \tilde{v}_l$ and $v_{s+2} = \tilde{v}_k$.)

(Exclusive) Neighborhood Sets

- For $i < j$, v_i and v_j are *exclusive neighbors* iff they are connected and, for all $k < i$, v_k and v_j are not connected.
- By our enumeration, for all $j > 0$, there is a unique $i < j$ where v_i and v_j are exclusive neighbors.
- At times we might label our nodes as $v_i = v_{n_i,i}$ where v_i and v_{n_i} are exclusive neighbors.
- The *(exclusive) neighborhood set* of v_i, $\mathcal{N}_i$, is all the exclusive neighbors of v_i.
- All neighbor sets are anticliques. So no copy of $\mathcal{H}$ lives in a neighbor set.
- These neighbor sets are dependance on our chosen fixed enumeration. When referring to a neighbor set we always refer to this enumeration.

$\mathcal{H}$ is indivisible

Theorem (Komjáth and Rödl)

Color the nodes of $\mathcal{H}$ with finitely many colors. Then there is a copy of $\mathcal{H}$ where all nodes have the same color (i.e. a homogenous copy).

Rephrase this to as nodes have finite big Ramsey degree in $\mathcal{H}$.
More precisely nodes have exact big Ramsay degree of 1 in $\mathcal{H}$.

Proof sketch in a few slides.

Any copy of $\mathcal{H}$

needs infinite many neighborhoods sets

Lemma

No subcopy of $\mathcal{H}$ (inside $\mathcal{H}$) lives in finitely many neighbor sets.

Proof.

Just color the each of the finitely many neighbor sets a different color. But no copy of $\mathcal{H}$ lives in a neighbor set. □

Large neighborhood sets in copies of $\mathcal{H}$

Lemma (large local neighborhood lemma)

For any subcopy $\tilde{\mathcal{H}} = (\tilde{V}, E)$ of $\mathcal{H}$, we can always find large l and a node v such that there are infinitely nodes in $\tilde{\mathcal{H}}$ connected to v and in $\mathcal{N}_l$ (hence $\mathcal{N}_l \cap \tilde{V}$ is infinite).

Proof.

Choose k large. Color $\mathcal{N}_j$ RED iff $j < k$. Now apply the indivisible result to get a BLUE copy. Take the first element, $v_{n_i,i}$, in our enumeration of $\mathcal{H}$ in this BLUE copy. In the BLUE homogenous copy there must be infinitely many nodes v connected to $v_{n_i,i}$. For each v there is a l, $k \leq l \leq n_i$, such that $v \in \mathcal{N}_l$. Hence for some such l, $\mathcal{N}_l$ is infinite. □

We can find copies of $\mathcal{H}$ where, for infinitely many i, $\mathcal{N}_i \cap \tilde{V}$ is nonempty but finite.

A large anticlique lives in every copy of $\mathcal{H}$

Lemma

For any subcopy $\tilde{\mathcal{H}} = (\tilde{V}, E)$ of $\mathcal{H}$, we can enumerate an infinite anticlique where each node of the anticlique lives in a unique neighborhood. Moreover during our enumeration we can have the gaps between a node and the next neighborhood set be large. I.e. the gap between i and n_{i+1} is large. Call this a large anticlique.

A neighborly coloring

- Color i RED iff the red nodes are *dense* in $\mathcal{N}_i$.
- I.e. for all finite disjoint $D, F \subset V$, where $i \in D$, D is an anticlique, and, for all $j < i, j \in F$, there is a RED node (in $\mathcal{N}_i$) connected to all nodes in D and disconnected from all nodes in F.
- Otherwise color i BLUE and in which case there are finite disjoint $D_i, F_i \subset V$, where $i \in D_i$, D_i is an anticlique, and, for all $j < i, j \in F_i$, where all nodes v which connected everything in D_i and disconnected everything in F_i are BLUE.

A neighborly proof of indivisibility

- Use pigeonhole principle to get infinitely many neighbor sets of the same color.
- If RED repeat our construction of $\mathcal{H}$ within these neighbor sets.
- But if BLUE in addition we must thin further to ensure the neighbor sets of our nodes are pairwise disjoint.

Two more results about indivisibility

Lemma (Upper bound)

If our coloring is computable we can find a homogenous copy in $\mathbf{0}''$*. Can be improved to* $\mathbf{0}'$*.*

Lemma (Gill, Lower bound)

There is a coloring with no computable homogenous copy.

Proof.

Wait till $W_{e,s}$ contains a large anticlique which cannot be contained in the first k neighbor sets where k is determined by (finite) priority (Gill uses a result of Folkman rather the large anticlique). Then color the remaining neighbor sets of $W_{e,s}$ so that W_e cannot be a homogenous copy of $\mathcal{H}$. □

Questions about indivisibility

Question (Cone avoidance)

Given a finite computable coloring the nodes of $\mathcal{H}$ and a non computable X is there a homogenous copy of $\mathcal{H}$ which does not compute X.

Question (Zoo)

Where is the zoo does this principle live?

$\mathcal{B}$ has finite big Ramsey degree in $\mathcal{H}$

Let $\mathcal{B}$ be a finite triangle free graph.

Theorem (Dobrinen)

There is an $\ell_{\mathcal{B}}$ such that when all copies of $\mathcal{B}$ within $\mathcal{H}$ are colored with finitely many colors, there is a copy of $\mathcal{H}$, $\tilde{\mathcal{H}}$, where the copies of $\mathcal{B}$ need at least $\ell_{\mathcal{B}}$ colors. We call $\tilde{\mathcal{H}}$ a minimal heterogeneous copy of $\mathcal{H}$ w.r.t. $\mathcal{B}$ (and the coloring). Moreover, there is a finite coloring of the copies of $\mathcal{B}$ such that all subcopies of $\mathcal{H}$ retain all $\ell_{\mathcal{B}}$ colors.

Dobrinen used very strong set existence axioms and forcing standard for set theory. A different proof (Hubicka) of the the result used the Carlson-Simpson theorem to compute a minimal heterogeneous copy. But in both cases nonarithmetical bounds were needed.

Arithmetical Upper Bounds

Theorem (Cholak, Dobrinen, Towsner)

There is a computable function $u(\mathcal{B})$ (on naturals) such that for every coloring of the copies of $\mathcal{B}$ there is a minimal heterogeneous copy w.r.t. $\mathcal{B}$ below $\mathbf{0}^{\mathbf{u}(\mathcal{B})}$.

The proof is an finite inductive (over the types) forcing proof like the above forcing proof for nodes. In fact, the node case is the last step of the induction. The pivot between colors needs Dobrinen's Halpern-Laüchli for coding trees but again with an arithmetic proof.

Arithmetical Lower Bounds

Theorem (Cholak, Dobrinen, McCoy)

There is computable eventually increasing function $l(\mathcal{B})$ and a coloring $c(\mathcal{B})$ such that if $|\mathcal{B}| \geq 2$ then $l(\mathcal{B}) > 0$ and every minimal heterogeneous copy w.r.t. $\mathcal{B}$ and $c(\mathcal{B})$ computes $\mathbf{0}^{\mathbf{l}(\mathcal{B})}$.

Proof adapts similar colorings of Jockusch.

Question

Can these results be improved to get the same function?

The $\ell_{\mathcal{B}}$ many types

The $\ell_{\mathcal{B}}$ types come from the interaction between the model $\mathcal{H}$ and our fixed enumerations. More than just the neighbor sets are definable but that additional structure will not be discussed here. These types have to be persistent in all subcopies of $\mathcal{H}$ but also in all possible subcopies (like what we have shown for the large anticlique). In the minimal heterogeneous copy every type gets just one color.

Some examples of types

- Just a node.
- Large anticlique of size n, $\{v_0, v_1, \ldots, v_{n-1}\}$ where $i < n_{i+1}$.
- An edge between two nodes v_i and v_j. Then $n_i \neq n_j$ (the nodes cannot be in same neighbor sets). Assume $n_i < n_j$. Then $n_j < \min i, j$ (the 2nd neighborhood set must start before the least node). But the relation between i and j is undetermined. Two options: $i < j$ (2nd node is higher) or $i > j$ (first node is higher). The 2nd option will be discussed later.

Large anticlique of size k

have exact big Ramsey degree of 1 in $\mathcal{H}$ implies Ramsey theorem for sets of size k.

Take given a coloring of $[\omega]^k$ and use it color the corresponding k-tuples of neighborhood sets. This induces a coloring on the large anticlique of size k. Now we get a homogenous copy for this type and coloring. The set of indexes for the neighborhood sets form a homogenous set for an initial coloring of $[\omega]^k$.

Edges

Consider the edge type where $i > j$ as above. Color these types using Jockusch's halting set coloring where $x = n_j, s = j$, and $t = i$. We can show any homogenous copy, $\tilde{\mathcal{H}}$, for this type and coloring must compute the halting set.

Edges persistent

We will need to apply the large local neighborhood lemma several times. First to get $v_{n_0,0}$. Now color all the nodes in our copy in neighborhoods below $n_0 + 1$ RED and the rest BLUE. We get a new BLUE homogenous copy inside our copy. Again apply the lemma to get $v_{n_1,1}$ in both our copies. Now ignore this new BLUE copy and work within our old copy. To get the gap large between x and s look for a node $v_{n_j,j}$ in our copy where $n_j \geq n_1$, n_j is large, and the gap between n_j and j is large. Such a node exists by the large local neighborhood lemma.

Now work with a $v_{n_j,j}$, where $n_j \geq n_1 > n_0$. $v_{n_0,0}$ and $v_{n_j,j}$ are not connected (otherwise $n_j \leq n_0$). By the extension property in our copy, there a large node v_i connected to both and not connected other nodes below some large node. $n_j \leq n_0 \leq n_j$ as desired.