Generically Computable Linear Orderings

David Gonzalez

U.C. Berkeley
Based on joint work with Wesley Calvert, Douglas Cenzer and Valentina Harizanov

April 2024
University of Wisconsin, Milwaukee

Summary of the talk

We are going to discuss *generic computability* in the context of computable structure theory. We will take a structural perspective and consider *Ramsey-like theorems.*

1. The α-Ramsey property
2. Results for linear orderings
3. Connection with generic and coarse computability

Ramsey's Theorem - Undergraduate Version

- Every infinite graph, has a subgraph isomorphic to K_∞ or the edgeless K_∞^c.
- Every infinite graph with finitely many colors has a monochromatic subgraph isomorphic to K_∞.
- Every infinite linear ordering has a sub-ordering isomorphic to ω or ω^*.
- Every infinite partial ordering has a sub-ordering isomorphic to ω, ω^* or an anti-chain.
- Every infinite equivalence structure has an infinite class or infinitely many classes.

Ramsey's Theorem abstractly

There are *many* ways to abstract Ramsey's theorem; the way below is tied tightly to the "undergraduate" examples enumerated:

Every countable structure in the class $\mathbb{K}$ has a substructure that is among the subclass $\mathbb{J} \subset \mathbb{K}$ of "simple" structures.

Ramsey's Theorem abstractly

There are *many* ways to abstract Ramsey's theorem; the way below is tied tightly to the "undergraduate" examples enumerated:

Every countable structure in the class $\mathbb{K}$ has a substructure that is among the subclass $\mathbb{J} \subset \mathbb{K}$ of "simple" structures.

It is often natural and useful to ask for more than a substructure. We may need to relax our notion of simple.

Notions of substructure

There is a natural stratified notion of substructure given to us by syntax: Σ_α-elementary substructures.
Definition: A substructure $\mathcal{A} \subseteq \mathcal{B}$ is Σ_α-elementary if for all $\bar{p} \in \mathcal{A}$ and $\psi \in \Sigma_\alpha$

$$\mathcal{A} \models \psi(\bar{p}) \iff \mathcal{B} \models \psi(\bar{p}).$$

Note: Σ_α is a set of $\mathcal{L}_{\omega_1,\omega}$ formulas with $\alpha \in \omega_1$. Our results will also hold for computably infinitary formulas or first order formulas at finite levels.

The α-Ramsey property

Definition: We say a class of structures $\mathbb{K}$ has the α-Ramsey property if:

Every countably infinite structure in the class $\mathbb{K}$ has a Σ_α-elementary substructure that is among the subclass $\mathbb{J} \subset \mathbb{K}$ of computably presentable structures.

Note: Ramsey's theorem gives that any class of relational structures has the 0-Ramsey property.

Focusing in

Results on linear orderings

The 1-Ramsey property

Theorem: [CCGH] The class of linear orderings has the 1-Ramsey property. In fact, every linear ordering has a Σ_1-elementary substructure isomorphic to ω, ω^*, ζ, $\omega \cdot \omega^*$, $\omega^* \cdot \omega$ or η.

The 2-Ramsey property

1. The set of *scattered* linear orderings has the 2-Ramsey property.
2. They must be in the ω,ω^*, ζ, $\omega \cdot \omega^*$ or $\omega^* \cdot \omega$ case from the previous theorem.
3. Each of these Σ_1-elementary substructures can be upgraded to a Σ_2-elementary substructure by adding in the first and last 1-block if they exist.

The 2-Ramsey property: a counterexample

1. The 2-Ramsey property does not hold for the class of all linear orderings.
2. The *shuffle sum* of countably many linear orderings $Sh(\{L_i\}_{i \in \omega})$ is obtained by partitioning η into countably many mutually dense sets and replacing all elements in the i^{th} part with a copy of L_i.
3. If A is a set of natural numbers, we let $Sh(A)$ be the shuffle sum where we treat each $n \in A$ as the unique linear orderings of size n.
4. $Sh(A)$ does not always have a Σ_2-elementary substructure with a computable copy, e.g. if A is Σ^0_3 immune.

A 2-Ramsey Question

Question: Can we describe the maximal class of linear orderings for which the 2-Ramsey property holds?

Rephrased: Can we describe which linear orderings have infinite, computable Σ_2-elementary substructures?

Rephrased again: Can we describe which linear orderings have Σ_2-generically c.e. copies?

Answer: No... at least not in a more efficient manner than the above descriptions.

Some Theorems

Theorem:[CCGH] The maximal class of linear orderings for which the 2-Ramsey property holds is $\mathbf{\Sigma}_1^1$-complete.

Question: Can we avoid this at higher levels?

Theorem:[CCGH] The maximal class of linear orderings for which the $(\alpha + 2)$-Ramsey property holds is $\mathbf{\Sigma}_1^1$-complete for any $\alpha \in \omega_1^{ck}$.

Note: This is new behavior for structures in general; these results also apply to structurally complete classes like graphs and groups.

Spilling the beans

Connections to Generic and Coarse Computability

Generic and Coarse Computability

Definition: Let $S \subseteq \mathbb{N}$.

1. The density of S up to n, denoted by $\rho_n(S)$, is given by

$$\rho_n(S) = \frac{|S \cap \{0, 1, 2, \ldots, n\}|}{n+1}.$$

2. The asymptotic density of S, denoted by $\rho(S)$, is given by $\lim_{n\to\infty} \rho_n(S)$.

A set A is said to be *generically computable* if and only if there is a partial computable function ϕ such that ϕ agrees with χ_A throughout the domain of ϕ, and such that the domain of ϕ has asymptotic density 1.

A set A is said to be *coarsely computable* if and only if there is a *total* computable function ϕ that agrees with χ_A on a set of asymptotic density 1.

Generically and Coarsely Computable structures

- $\mathcal{A}$ is Σ_α-*generically c.e.* if there is a dense c.e. set D such that the substructure $\mathcal{D}$ with universe D is a c.e. substructure and also a Σ_α elementary substructure of $\mathcal{A}$.
- A structure $\mathcal{A}$ is Σ_α-*coarsely c.e.* if there exist a dense set D and a c.e. structure $\mathcal{E}$ such that the substructure $\mathcal{D}$ with universe D is a Σ_α elementary substructure of both $\mathcal{A}$ and $\mathcal{E}$.

Generically and Coarsely Computable copies

Lemma: For a linear ordering $\mathcal{A}$, $\mathcal{A}$ has a Σ_α-generically c.e. copy if and only if it has a Σ_α elementary substructure that is isomorphic to a computable structure.

Rephrased results

We get the following theorems using similar arguments to the Ramsey-like results:

Theorem:[CCGH] Every linear ordering has a Σ_1-generically c.e. copy.

Theorem:[CCGH] Every linear ordering has a Σ_1-coarsely c.e. copy.

Theorem:[CCGH] The class of linear orderings with a $\Sigma_{\alpha+2}$-generically c.e. copy is $\boldsymbol{\Sigma}^1_1$-complete.

Theorem:[CCGH] The class of linear orderings with a $\Sigma_{\alpha+2}$-coarsely c.e. copy is $\boldsymbol{\Sigma}^1_1$-complete.

A contrast with normal computability

This shows that generic and coarse computability act very differently to ordinary computability.

Theorem:[Nadel] The set of models with computable copies in any $\mathcal{L}_{\omega_1,\omega}$ elementary class is Borel. If the class is an $\mathcal{L}_{c,\omega}$ elementary class, then this set is $\mathbf{\Sigma}^0_{\omega_1^{ck}+3}$ at worst.

We even see this difference at the lowest interesting level of our hierarchy:
Theorem:[CCGH] The class of successor linear orderings with a Σ_1-generically c.e. copy is $\mathbf{\Sigma}^1_1$-complete.

Thank you!