Strong minimal pairs in the enumeration degrees

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1. The $\exists\forall$ theory of degree structures

2. No strong super minimal pairs

3. Strong minimal pairs

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Motivating question

Question

At what level of quantifier complexity does the theory of a degree structure become undecidable?

- For $\mathcal{D}_T$ we know that the $\exists\forall$ theory is decidable, but the $\exists\forall\exists$ theory in undecidable.
- For the c.e. Turing degrees we know the $\exists$ theory is decidable and the $\exists\forall\exists$ theory is undecidable but do not know about the $\exists\forall$ theory.

What is known for $\mathcal{D}_e$

Theorem (Lagemann '72)

Every finite lattice embeds into the enumeration degrees. Hence the $\exists$ theory is decidable.

Theorem (Kent '06)

The $\exists\forall\exists$ theory of $\mathcal{D}_e$ is undecidable.

Generalized extension of embeddings

It turns out that the $\exists\forall$ theory of a partial order is equivalent to the following question.

Question (Generalized extension of embeddings)

Given finite partial orders $\mathcal{P}$ and $\mathcal{Q}_0, \dots, \mathcal{Q}_{k-1}$ is it true that every embedding of P into $\mathcal{D}$ can be extended to $\mathcal{Q}_i$ for some $i < k$?

The case when $k = 1$ is known as the extension of embedding problem. Lempp, Slaman and Soskova, '21 proved that the extension of embeddings problem is decidable for the e-degrees. via the following theorem

Theorem (Lempp, Slaman, Soskova '21)

Every finite lattice embeds into the enumeration degrees a strong interval.

Example questions

The speaker will now draw some lattices on the board.

Types of minimal pairs

Definition

In an upper semilattice with least element 0 a pair $a, b > 0$ is a:

- *minimal pair* if $a \wedge b = 0$.
- *strong minimal pair* if it is a minimal pair, and for all x such that $0 < x \leq a$ we have $x \vee b = a \vee b$.
- *super minimal pair* if both a, b and b, a are strong minimal pairs.
- *strong super minimal pair* if it is a minimal pair, and for all x, y such that $0 < x \leq a$ and $0 < y \leq b$ we have $x \vee y = a \vee b$.

What is now known

Theorem (J-G, Soskova)

There are no strong super minimal pairs in the enumeration degrees.

Theorem (J-G/Anonymous referee)

There are strong minimal pairs in the enumeration degrees.

Question

Are there super minimal pairs in the enumeration degrees?

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Enumeration reducibility

Definition

We define $A \leq_e B$ if is a c.e. set of axioms W such that

$$x \in A \iff \exists \langle x, u \rangle \in W[D_u \subseteq B]$$

where $(D_u)_u$ is a listing of all finite sets by strong indices.

- We have that $\leq_e$ is a pre-order and taking equivalences classes give us a degree structure $\mathcal{D}_e$.
- The lowest element of $\mathcal{D}_e$ is 0_e which is the class of c.e. sets.
- The Turing degrees embed into $\mathcal{D}_e$ as a definable substructure.
- From an effective listing of c.e. sets $(W_e)_e$ we obtain an effective listing of enumeration operators $(\Psi_e)_e$. Defined by $A = \Psi_e(B)$ if $A \leq_e B$ via the set of axioms W_e.
- Unlike Turing operators $\Psi_e(A)$ is always a set. We also have that these operators are monotonic: if $B \subseteq A$ then $\Psi_e(B) \subseteq \Psi_e(A)$.

The Gutteridge operator

Theorem (Gutteridge '71)

For every $\mathbf{a} \neq \mathbf{0}_e$ *there is* $\mathbf{b} \in \mathcal{D}_e$ *such that* $\mathbf{0} < \mathbf{b} < \mathbf{a}$.

AS part of his proof, Gutteridge constructed an enumeration operator Θ with the following properties:

1. If A is not c.e. then $\Theta(A) <_e A$.
2. If $\Theta(A)$ is c.e. then A is Δ^0_2.

No strong super minimal pairs outside of Δ^0_2

The construction of Θ produces a sequence $(n_k)_k$ such that:

- $B = \bigoplus_k n_k$ is a c.e. set.
- $\Theta(A) = B \cup \{\langle k, n_k \rangle : k \in A\}$.

Lemma

$\Theta(A \cup C) = \Theta(A) \cup \Theta(C)$.

Lemma (J-G)

If A and C are not Δ^0_2 then there are X, Y such that $\emptyset <_e X \leq_e A$, $\emptyset <_e Y \leq_e C$, and $X \oplus Y <_e A \oplus C$.

Proof.

Take $X = \Theta(A \oplus \emptyset)$, $Y = \Theta(\emptyset \oplus C)$. □

$\mathcal{K}$-pairs

Definition (Kalimullin '03)

A and B are a Kalimullin pair ($\mathcal{K}$-pair) if there is a c.e. set $W \subseteq \omega^2$ such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$. A $\mathcal{K}$-pair is called *trivial* if one of A, B is c.e.

Kalimullin pairs have been used to prove that the jump is definable in $\mathcal{D}_e$ (Kalimullin '03) and that the total degrees are definable (Ganchev and Soskova '15).

No strong minimal with A in Δ^0_2

We use the following two facts about $\mathcal{K}$-pairs.

Theorem (The minimal pair $\mathcal{K}$-property, Kalimullin '03)

A, B are a $\mathcal{K}$-pair if and only if for all $X \subseteq \omega$, $A \oplus X$ and $B \oplus X$ form a minimal pair relative to X. i.e. $Y \leq_e A \oplus X, Y \leq_e B \oplus X \implies Y \leq_e X$.

Theorem (Kalimullin '03)

Every nonzero Δ^0_2 degree computes a nontrivial $\mathcal{K}$-pair.

Theorem (Soskova)

If A is Δ^0_2 then A, B is not a strong minimal pair in $\mathcal{D}_e$ for any B.

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$\mathcal{K}$-pair construction

Theorem (Anonymous referee)

If A, B are a non trivial $\mathcal{K}$-pair with $B \leq_e \emptyset'$ and $A \not\leq_e \emptyset'$, then $(A, \emptyset')$ form a strong minimal pair.

Forcing construction

Idea for dealing with one operator Ψ.

- Build functions $H : 2^{<\omega} \to \mathcal{P}_{\text{fin}}(\omega)$ and $S : 2^{<\omega} \to \omega$and such that for all σ:
  - $S(\sigma) \in \Psi(H(\sigma) \cup \bigcup_{k:\sigma(k)=1} H(\sigma \restriction k))$.
  - $S(\sigma) \notin \Psi(\omega \setminus H(\sigma) \cup \bigcup_{k:\sigma(k)=0} H(\sigma \restriction k))$.
- Do a second round of forcing to construct a generic path X.
  $A = \bigcup_{k:X(k)=1} H(X \restriction k)$ and B codes

Thank you

Thank You