Infinitary logic has no expressive efficiency over finitary logic

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Infinitary Logic I

Infinitary languages within $L_{\infty,\omega}$ allow conjunctions and disjunctions over infinite sets of formulas. This allows us to express concepts not expressible in finitary first order logic.

The torsion groups are the groups which are models of

$$\forall x \bigvee_{n} x^n = e$$

The finitely generated abelian groups are those which are models of

$$\bigvee_{n} \exists x_1 \ldots \exists x_n \forall y \bigvee_{(k_1,\ldots,k_n)\in\mathbb{Z}^n} \left(y = \sum_{i\leq n} k_i \cdot x_i \right)$$

Infinitary Logic II

Within countable structures, the language $L_{\omega_1,\omega}$ allowing countably infinite conjunctions and disjunctions is powerful enough to completely specify a structure's isomorphism type.

Theorem (Scott 1965)

For a countable structure $\mathcal{A}$, there is a sentence $\varphi \in L_{\omega_1,\omega}$, such that the countable models of φ are exactly the structures isomorphic to $\mathcal{A}$.

$L_{\omega_1,\omega}$ also arises when one considers the descriptive set theory of countable structures.

Theorem (Lopez-Escobar 1965)

If $\mathfrak{X}$ is an isomorphism invariant Borel set of countable structures, it is the set of models of a sentence of $L_{\omega_1,\omega}$ (and conversely).

Complexity of Formulas I

We can measure the complexity of infinitary formulas by counting alternations of quantifiers, as well as infinite conjunctions and disjunctions. This gives rise to the classes $\Sigma_\alpha, \Pi_\alpha \subset L_{\omega_1,\omega}$.

Theorem (Vaught 1975)

If $\mathfrak{X}$ is an isomorphism invariant $\mathbf{\Pi}^0_\alpha$ ($\mathbf{\Sigma}^0_\alpha$) set of countable structures, it is the set of models of a Π_α (Σ_α) sentence of $L_{\omega_1,\omega}$

Theorem (Ash–Knight–Manasse–Slaman 1989, Chisholm 1990)

If R is an extra relation on a countable structure $\mathcal{A}$ that is relatively intrinsically Π^0_α (Σ^0_α) in isomorphic copies of $\mathcal{A}$, then R is defined in $\mathcal{A}$ by a computable Π_α (Σ_α) formula of $L_{\omega_1,\omega}$.

Complexity of Formulas II

We will take a somewhat coarser perspective and count only quantifiers, giving rise to classes $\forall_\alpha$, $\exists_\alpha$. Because our interest is in comparing the complexity of infinitary and finitary formulas, we will consider $\forall_n, \exists_n$, for n finite.

The sentence

$$\bigvee_n \exists x_1 \dots \exists x_n \forall y \bigvee_{(k_1,\dots,k_n)\in\mathbb{Z}^n} \left(y = \sum_{i \le n} k_i \cdot x_i \right)$$

defining finitely generated abelian groups is Σ_3, but $\exists_2$.

Note that $\Pi_n \Rightarrow \forall_n$ and $\Sigma_n \Rightarrow \exists_n$, so our results for $\forall_n$ and $\exists_n$ infinitary formulas include Π_n and Σ_n formulas.

The Theorem

Theorem (Keisler 1965, Harrison-Trainor–K 2023)

Suppose T is a finitary theory, $\varphi(\overline{x})$ is a finitary formula, and $\psi(\overline{x})$ is an infinitary $\forall_n$ ($\exists_n$) formula such that φ and ψ are equivalent in all models of T. Then there is a finitary $\forall_n$ ($\exists_n$) formula θ such that φ, ψ, and θ are equivalent in all models of T.

Thus in spite of its greater expressive power, $L_{\infty,\omega}$ cannot express first order concepts in a more efficient or simple way than finitary first order logic.

The intersection of the properties expressible in finitary first order logic and the properties expressible by a $\forall_n$ infinitary formula is exactly the properties expressible by a finitary $\forall_n$ formula.

Refinements

1. In the preceding theorem, we may take θ such that any symbol occurring positively (negatively) in θ already occurs positively (negatively) in ψ.

2. If instead of a single first order formula, ψ is equivalent to a first order theory T, then T has an axiomatization by sentences θ_i whose quantifier complexity is bounded by that of ψ and satisfy the condition described above.

Application

Theorem (Andrews–Gonzalez–Lempp–Rossegger–Zhu 2024)

Let T be a first order theory in a countable language. The following are equivalent.

1. *T has a $\forall_n$ axiomatization.*
2. *The set of countable models of T is $\mathbf{\Pi}^0_n$.*

Proof.

One direction is straightforward, a $\forall_n$ axiomatization directly gives a $\mathbf{\Pi}^0_n$ description of the set of countable models.

For the other direction, first apply Vaught's theorem to get a $\forall_n$ sentence of $L_{\omega_1,\omega}$ equivalent to T, then conclude that T has a $\forall_n$ axiomatization. □

Forcing I

Our proofs of these theorems are based on a notion of forcing with elementary extensions.

Infinitary formulas can change truth value in elementary extensions, and the forcing relations helps us keep track of what can happen in a further elementary extension.

The sentence

$$\psi = \forall x \left(Q(x) \to \exists y \left(R(x,y) \wedge \bigwedge_n \neg P_n(y) \right) \right)$$

can switch between true and false infinitely often as one goes up elementary extensions of a certain structure $\mathcal{A}$. It turns out that $\mathcal{A}$ forces ψ.

Forcing II

This is similar in spirit to Robinson's infinite model theoretic forcing, with two main differences

1. Where Robinson forcing uses arbitrary extensions of structures, we restrict ourselves to elementary extensions, so first order properties are preserved
2. We define a forcing relation for formulas in $L_{\infty,\omega}$.

This notion of forcing is thus suited to study the relationship between finitary and infinitary logic.

Forcing III

The forcing machinery consists of strong and weak forcing relations

$$\mathcal{A} \Vdash \psi(\bar{a})$$

$$\mathcal{A} \Vdash^{\star} \psi(\bar{a})$$

for A an L-structure, $\bar{a} \in A$ and $\psi(\bar{x})$ a formula in $L_{\infty,\omega}$.

$\Vdash$ is defined by recursion on the structure of ψ, weak forcing can be defined by

$$\mathcal{A} \Vdash^{\star} \psi(\bar{a}) \Leftrightarrow \mathcal{A} \Vdash \neg\neg\psi(\bar{a})$$

Strong forcing propagates up elementary extensions. Due to elementary amalgamation, the behavior of strong and weak forcing is simplified in certain ways. In particular, weak forcing is preserved in both directions by elementary extensions.

Forcing IV

To extract useful information from the forcing relation, we show that for any $\mathcal{A}$ and any fragment $F \subset L_{\infty,\omega}$, $\mathcal{A}$ has an F-**generic** elementary extension $\mathcal{G}$.

This $\mathcal{G}$ satisfies

$$\mathcal{G} \Vdash \psi(\bar{a}) \Leftrightarrow \mathcal{G} \Vdash^{\star} \psi(\bar{a}) \Leftrightarrow \mathcal{G} \models \psi(\bar{a})$$

for any $\psi \in F$. Forcing is equivalent to truth here.

Thus if ψ is already preserved by elementary extensions (for instance, if it is equivalent to a finitary formula) then ψ is true in $\mathcal{A}$ if and only if it is weakly forced in $\mathcal{A}$. We can check its truth by passing to a generic extension.

Definability I

Because weak forcing propagates up and down elementary extensions, whether

$$\mathcal{A} \Vdash^{\star} \psi(\bar{a})$$

depends only of the (finitary) type of $\bar{a}$ in $\mathcal{A}$. This lets us define weak forcing by a disjunction over types.

In fact, by induction of the structure of ψ, one can show that weak forcing for ψ can be defined by a formula

$$\mathrm{Force}_{\psi} = \bigvee\!\!\!\bigvee_{\alpha} \bigwedge\!\!\!\bigwedge_{\beta} \theta_{\alpha,\beta}$$

where the $\theta_{\alpha,\beta}$ have quantifier complexity no greater than ψ. The $\theta_{\alpha,\beta}$ are obtained by rearranging the quantifiers and infinite connectives of ψ.

Definability II

If ψ is preserved by elementary extensions, ψ is equivalent to Force_ψ.

Thus if ψ is equivalent to a finitary formula φ, φ is equivalent to Force_ψ.

If ψ is $\forall_n$, one can use this to show that φ is equivalent to a finitary $\forall_n$ formula, by compactness.

$$\varphi \longleftrightarrow \bigvee\!\!\!\!\bigvee_{\alpha} \bigwedge\!\!\!\!\bigwedge_{\beta} \theta_{\alpha,\beta}$$

$$\varphi \longleftrightarrow \bigvee\!\!\!\!\bigvee_{\alpha} \bigwedge_{\beta \in B_\alpha} \theta_{\alpha,\beta}$$

$$\varphi \longleftrightarrow \bigvee_{\alpha \in A} \bigwedge_{\beta \in B_\alpha} \theta_{\alpha,\beta}$$

Other Approaches I

We recently learned that this theorem is a consequence of a result in Keisler's *Finite Approximations of Infinitely Long Formulas*.

Keisler defines a notion of a finite approximation of an infinitary formula, recursively on the structure of the formula in a manner analogous to our formula Force_ψ.

He then proves by induction on the structure of the formula that sufficiently saturated models which are distinguished by an infinitary formula are distinguished by one of its finite approximations.

This implies that if an infinitary formula is equivalent to a finitary formula, it is equivalent to a positive boolean combination of its finite approximations.

Other Approaches II

Part of the statement for $L_{\omega_1,\omega}$ follows from an effective descriptive set theoretic analysis of the set of countable models of a first order theory in a recent paper from Andrews–Gonzalez–Lempp–Rossegger–Zhu.

They prove directly that if a theory has no $\forall_n$ axiomatization, its set of models is $\mathbf{\Sigma}^0_n$-hard. Therefore, a theory has a $\forall_n$ axiomatization if and only if its set of models is $\mathbf{\Pi}^0_n$.

Using the fact that the set of models of a Π_n sentence of $L_{\omega_1,\omega}$ is $\mathbf{\Pi}^0_n$, and considering the theory $\{\varphi\}$, one concludes that if φ is equivalent to a Π_n formula of $L_{\omega_1,\omega}$, it is equivalent to a finitary $\forall_n$ formula.

Thank You!

Theorem (Keisler 1965, Harrison-Trainor–K 2023)

Suppose T is a finitary theory, $\varphi(\overline{x})$ is a finitary formula, and $\psi(\overline{x})$ is an infinitary $\forall_n$ ($\exists_n$) formula such that φ and ψ are equivalent in all models of T. Then there is a finitary $\forall_n$ ($\exists_n$) formula θ such that φ, ψ, and θ are equivalent in all models of T.