Computability theory of operator algebras

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- Classical computable structure theory studies countable algebraic and combinatorial structures such as rings, fields, groups, graphs, linear orders, etc.
- Missing from this list: the structures of analysis such as Banach spaces, Hilbert spaces, C^* algebras, etc. These are called *metric structures*.
- The model theory of metric structures has been extensively developed by Henson, Ben-Yaacov, et. al.
- The program of *effective metric structure theory* uses the tools of computable analysis to extend the realm of computable structure theory to separable metric structures.

In this talk, I will focus on recent work concerning C^* algebras. Let's recall what these are.

Definition

A C^* *algebra* is a complete normed $*$-algebra A (over $\mathbb{C}$) so that $\|uu^*\| = \|u\|^2$ for every vector u of A.

Example

$\mathrm{Mat}_{n,n}(\mathbb{C})$ where the involution is the adjoint.

Example

$C^*([0,1])$.

The second of these examples can be generalized as follows.

Definition

When X is a compact Polish space, $C^*(X)$ is the C^* algebra consisting of the continuous functions from X into $\mathbb{C}$. The involution is given by pointwise conjugation.

Why C^* algebras?

- Because it's there.
- If A is a C^* algebra, then there is a Hilbert space H so that A is isometrically isomorphic to an algebra of bounded linear operators on H.
- C^* algebras important for quantum information theory.
- Model theory of C^* algebras recently experienced considerable growth.
- Computability of theories of C^* algebras plays a role in Goldbring and Hartt's refutation of the Connes Embedding Conjecture.

Let's start by defining what we mean by a computable presentation of a C^* algebra. (Definition due to A. Fox [**?**].)

Definition

Suppose A is a C^* algebra.

1. A *generating sequence* for A is a sequence $(v_n)_{n\in\mathbb{N}}$ of vectors of A that generates a dense $*$-algebra of A.
2. If $(v_n)_{n\in\mathbb{N}}$ is a generating sequence for A, then $(A, (v_n)_{n\in\mathbb{N}})$ is a *presentation of A*.
3. If $A^\# = (A, (v_n)_{n\in\mathbb{N}})$ is a presentation of A, then each vector in the $*$-algebra generated by $(v_n)_{n\in\mathbb{N}}$ is a *rational vector of* $A^\#$.
4. A presentation $A^\#$ is *computable* if the norm is computable on the rational vectors of A.

Some standard presentations

Example

$\mathbb{C}$: declare the n-distinguished vector to be 1.

Example

$C^*([0,1])$: declare the n-th distinguished vector to be the power function $t \mapsto t^n$.

Example

$\mathrm{Mat}_{n,n}(\mathbb{C})$: use the standard basis.

Remark

All of these presentations are computable.

An initial goal of this direction is to understand the effective content of the following classical result due to Gelfand.

Theorem

If A is a commutative unital C^* algebra, then there is a compact Polish space X so that A is isometrically isomorphic to $C^*(X)$.

To understand the effective content of this theorem, we first must consider computably compact presentations of Polish spaces which we define now.

Convention

Throughout the rest of this talk, X is a compact Polish space.

Definition

A *presentation* of X is a triple $(X, d, (p_n)_{n\in\mathbb{N}})$ where d is a metric that is compatible with X and $(p_n)_{n\in\mathbb{N}}$ is dense in X.

Definition

Suppose $X^\# = (X, d, (p_n)_{n\in\mathbb{N}})$ is a presentation of X. We call p_n the *n-th distinguished point of $X^\#$*.

Convention

Throughout the rest of this talk, $X^\#$ denotes a presentation of X.

Definition

Suppose $X^{\#}$ is a presentation of X.

1. $X^{\#}$ is *computable* if its metric is computable on its distinguished points.
2. $X^{\#}$ is *computably compact* if it is computable and if from $k \in \mathbb{N}$ it is possible to compute distinguished points $p_0, \ldots, p_n$ of $X^{\#}$ so that $X = \bigcup_j B(p_j; 2^{-k})$.

Theorem (Fox 2022+ [?])

If X has a computably compact presentation, then $C^*(X)$ is computably presentable.

Theorem (BEFGHMMT 2024+ [?])

If $C^*(X)$ has a computable presentation, then X has a computably compact presentation.

Remark

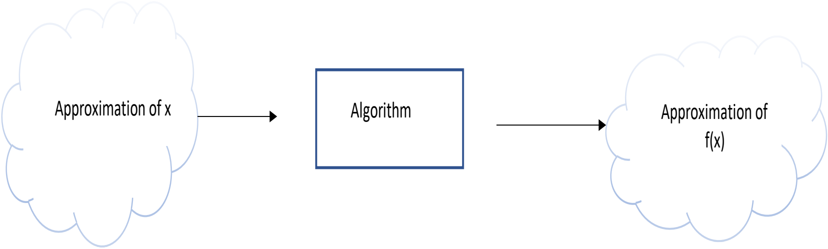
Both of the above theorems are *highly* uniform. Together, they classify the computably presentable C^* algebras that are unital and commutative. Our next step is to analyze the computable categoricity of these spaces. First, we will take a detour through *evaluative presentation*. The definition of these presentations require we first understand computability of maps between presented metric structures.

Foundations: how to think about computing functions on metric structures

To define what we mean by a computable function in the metric setting requires us to rethink our assumptions about what it means to compute a function. We usually think of a computable function like this:

Foundations: how to think about computing functions on metric structures

However, a point in a metric structure typically requires an infinite amount of information to be specified, so this model no longer makes sense. In practice, one replaces exact specifications with approximations:



- However, if our algorithm always produces the same approximation of $f(x)$, that's not very useful; we need a *convergence criterion*. That is, the machine can compute *arbitrarily good approximations* from *sufficiently good approximations*.
- Philosophy: an approximation is a rational open ball (ball whose center is a rational vector and whose radius is a positive rational real).
- There are numerous ways to formalize these intuitions; take your pick or come up with your own.

Definition

Let $X^\#$ be a presentation of a compact Polish space X, and let $C^*(X)^\#$ be a presentation of $C^*(X)$. We say that $C^*(X)^\#$ is *evaluative over* $X^\#$ if the evaluation map $(f, p) \mapsto f(p)$ of $C^*(X)$ is a computable map from $C^*(X)^\# \otimes X^\#$ to $\mathbb{C}$.

Here, $C^*(X)^\# \otimes X^\#$ is the presentation of the metric space $C^*(X) \times X$ induced by the presentations $C^*(X)^\#$ and $X^\#$. We have three key results on evaluative presentations.

Theorem (M. 2024+)

If $C^*(X)^\#$ is computable, then, up to computable homeomorphism, there is a unique presentation $X^\#$ over which $C^*(X)^\#$ is evaluative.

Theorem (M. 2024+)

If there is a computable presentation of $C^*(X)$ that is evaluative over $X^\#$, then $X^\#$ is computably compact.

Theorem (M. 2024+)

If $X^\#$ is computably compact, then, up to computable isometric isomorphism, there is a unique computable presentation of $C^*(X)$ that is evaluative over $X^\#$.

By means of these theorems, we obtain an effective version of the following.

Theorem (“Banach-Stone")

If T is an isometric isomorphism of $C^*(X)$ onto $C^*(Y)$, then there is a unique homeomorphism ψ of Y onto X so that $T(f) = f \circ \psi$ for all $f \in C^*(X)$.

Definition

ψ is called the *spatial realization of T*.

Theorem (M. 2024+)

Suppose for $j \in \{0, 1\}$ $C^*(X_j)^\#$ is computable and evaluative over $X_j^\#$. Further, suppose T is a computable isometric isomorphism of $C^*(X_0)^\#$ with $C^*(X_1)^\#$. Then, the spatial realization of T is a computable map of $X_1^\#$ to $X_0^\#$.

Corollary

TFAE.

1. *$C^*(X)$ is computably categorical.*
2. *Any two computably compact presentations of X are computably homeomorphic.*

Again, the proofs of the theorem and corollary are highly uniform.

From the corollary, we can generate some new examples.

Theorem (Fox 2022)

$C^*([0,1])$ is not computably categorical.

Corollary

There are two computably compact presentations of the Polish space $[0,1]$ that are not computably homeomorphic.

Theorem (M. 2024+)

Any two computably compact presentations of the Polish space 2^ω are computably homeomorphic.

Corollary (M. 2024+)

$C^*(2^\omega)$ is computably categorical.

By contrast:

Theorem (Thewmorakot 2023 [?])

The Banach space $C(2^\omega)$ is not computably categorical.

Melnikov and Ng recently claimed to have constructed a compact Polish space Y so that $C(Y)$ is computably presentable but Y does not have a computably compact presentation.

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