A topological approach to selection rules and stochasticity

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Stochasticity

- Basic idea: Stochastic sequences exhibit statistical properties that are invariant under selection of subsequences.
- Example: Church stochasticity – balanced limit frequency of 0s and 1s is preserved under computable selection rules.
- Separating stochasticity (and randomness) notions is mostly by "ad hoc" arguments.
- Q: Is there a more general approach that could shed some light on the structural differences of the various notions?

Stochasticity

- What can we say about stochasticity notions simply by looking at the topological properties of the selection rules?

- General framework:

    - A **selection rule** on a set X is a partial function $s :\subseteq X \to X$.
      A class S of selection rules is **monoidal** if

        - The identity function $\mathrm{id} : X \to X$ is in S,

        - S is closed under composition.

Selection rules

- **Id**: only identity mapping
- **Shift**: iterates of the shift operator on 2^ω
- **Fun**: all partial functions on 2^ω
- **Sub**: all infinite subsequences
- **Sub[D]**: all infinite subsequences of positive density
- **Sub[LD]**: all infinite subsequences of positive lower density
- **Sub[.][C]**: as above, but subsequences are required to be computable
- **Church**: computable functions $s : 2^{<\omega} \to \{0,1\}$
- **MWC**: partial computable functions $s :\subseteq 2^{<\omega} \to \{0,1\}$

Derivative and interior

- Let S be a monoidal class of selection rules on a set X.
- For each $x \in X$, let $V(x) = \{s(x) : s \in S\}$ (the ***derivative*** of x).
- For $Y \subseteq X$, let

$$V(Y) = \bigcup_{x \in X} V(x)$$

$$N(Y) = \{x \in X \colon V(x) \subseteq Y\}$$

- $N(Y)$ is called the ***interior*** of Y.

Selection Rule Topology

- **PROP:**

  1. Letting $\mathscr{T}$ be the set of all $Y \subseteq X$ for which $N(Y) = Y$ yields a topology on X.

  2. $(X, \mathscr{T})$ is an **Alexandroff space**, i.e. it is closed under *arbitrary* intersections. We denote it by X_S.

- In this topology,

  - $V(x)$ is the smallest open neighborhood of x.

  - $V(Y)$ is the smallest open set containing Y.

  - $N(Y)$ is the largest open set contained in Y.

Basic properties

- If S is countable and X is uncountable, then X_S is neither compact nor Lindelöf. However, regardless of the cardinality of S or X, $V(x)$ is compact for each x so X_S is locally compact.
- For "most" S, X_S is path-connected.
- *Preorder*: $y \leq x :\Leftrightarrow V(y) \subseteq V(x)$
  - Then $U \subseteq X$ is open iff U is closed downward under $\leq$.
  - $f : X_S \to Y_S$ is continuous iff f is monotone with respect to $\leq$.
  - f is a homeomorphism iff it is a $\leq$-order isomorphism.

Homotopy equivalence

- **THM:** Let X_S and $X_{S'}$ be two selection rule spaces with $S \subseteq S'$. Suppose there is a map $f: X_{S'} \to X_S$ satisfying the following two criteria:

  1. If $x, y \in X$ and there is some $s' \in S$ such that $s'(x) = y$, then there is some $s \in S$ such that $s(x) = y$.

  2. For all $x \in X$, there is some $s \in S$ such that either $s(x) = f(x)$ or $s(f(x)) = x$.

  Then X_S and $X_{S'}$ are homotopy equivalent.

  We call homotopy equivalences of this kind ***good***.

Homotopy equivalence

- The theorem follows from a general condition for homotopy equivalence for Alexandroff spaces via their preorders. The conditions (1) and (2) correspond to
    - For all $x, y \in X$ with $x \leq y$, $f(x) \leq' f(y)$
    - For all $x \in X$, either $x \leq' f(x)$ or $f(x) \leq' x$.
- The general characterization uses a characterization of homotopic maps between simplicial complexes due to P. May.

Homotopy equivalence

- Three different type of classifications for spaces $X_S, X_{S'}$ with $S \subseteq S'$:

1. There is a good homotopy equivalence between the two spaces.

2. The spaces are not homotopy equivalent.

3. No good homotopy equivalence exists between the two spaces (but they may still be homotopy equivalent).

Coarse and fine

- **PROP:** Let S be a monoidal class of selection rules on 2^ω. Suppose there is some x_0 such that for all $y \in 2^\omega$, there is an $s \in S$ such that either $s(x_0) = y$ or $s(y) = x_0$. Then 2^ω_S is homotopy equivalent to 2^ω_{Fun}.

- **COR:** The topologies induced by $\mathbf{Sub}$, $\mathbf{Sub[LD]}$, $\mathbf{Sub[D]}$ are all homotopy equivalent to 2^ω_{Fun}, the trivial topology.

- **PROP:** The topologies induced by $\mathbf{Id}$ and $\mathbf{Shift}$ are not homotopy equivalent to the topology induced by any other class of selection rules considered.

Subsequence spaces

- **PROP:** The topologies induced by the classes $\mathbf{Sub}[\mathrm{C}]$, $\mathbf{Sub}[\mathrm{C}][\mathrm{LD}]$, and $\mathbf{Sub}[\mathrm{C}][\mathrm{D}]$ are all (good) homotopy equivalent.
- **PROP:** There is no good homotopy equivalence between the topologies induced by
  - $\mathbf{Sub}[\mathrm{C}]$ and$\mathbf{Sub}$,
  - $\mathbf{Sub}[\mathrm{C}][\mathrm{LD}]$ and $\mathbf{Sub}[\mathrm{LD}]$,
  - $\mathbf{Sub}[\mathrm{C}][\mathrm{D}]$ and $\mathbf{Sub}[\mathrm{D}]$.

Stochasticity

- **THM:** The topologies induced by $\mathbf{Church}$ and $\mathbf{MWC}$ are homotopy equivalent.
  - Define an f which induces a good homotopy equivalence
  - Key: include all information about how every Turing functional Φ_e acts on X, including the use and halting time of these computations.

$$f(x)(\langle e, u, k\rangle) = \begin{cases} 1 & k > 0 \text{ and } \Phi_{e,k}(x|_u)\downarrow \\ 0 & k > 0 \text{ and } \Phi_{e,k}(x|_u)\uparrow \\ 1 & k = 0 \text{ and } \Phi_e(x|_u)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

Further investigations

- Fill in the remaining (non-) homotopy equivalences. For example, is $\mathbf{Church}$ homotopy equivalent to $\mathbf{Sub[C]}$?
- Consider notions finer than homotopy equivalence, e.g. *computable* homotopy equivalence, or in general, A-computable homotopy equivalence.
- Would the lack of a computable homotopy equivalence between the topologies induced by $\mathbf{Church}$ and $\mathbf{MWC}$ ensure that the Church and MWC stochastics are not the same set? If not, what other properties of the topologies are needed?

Further investigations

- Consider *topological* stochasticity:
  - A set $D \subseteq X$ is ***A-dense*** if $V(D) = X$.
  - D-stochastic: element of $N(D)$.
  - Observe: The set of balanced 0-1-sequences is A-dense in $2^{\omega}_{\mathrm{Church}}$.