Asymptotic Notions of Computability: Minimal Degrees and Minimal Pairs

Tiago Royer

University of Chicago

April 21st, 2024

https://cs.uchicago.edu/~royer/ams.pdf

Intuition

"Definition"

A Turing machine M **solves** a problem P
if **for every** instance x of P,
M halts on x with the correct answer.

"Definition"

A Turing machine M **asymptotically solves** a problem P
if **for almost every** instance x of P,
M halts on x with the correct answer.

Density Definition

Definition

A subset A of $\{0,1\}^*$ is **dense** if

$$\lim_{n\to\infty} \frac{|\{x \in A : |x| = n\}|}{2^n} = 1$$

and **sparse** if the limit is 0.

Sparsity is equivalent to

$$|\{x \in A : |x| = n\}| = o(2^n)$$

Coarse and Generic computability

Definition

A set A is **coarsely computable**
if there exists a Turing machine M such that
$M(x)\downarrow$ for all x and the set

$$\{x \mid M(x) = A(x)\}$$

is dense.

Definition

A set A is **generically computable**
if there exists a Turing machine M such that
$M(x)\downarrow$ implies $M(x) = A(x)$ and the set

$$\{x \mid M(x)\downarrow\}$$

is dense.

Examples

Example

Every computable set is both coarsely and generically computable.

Example

The set

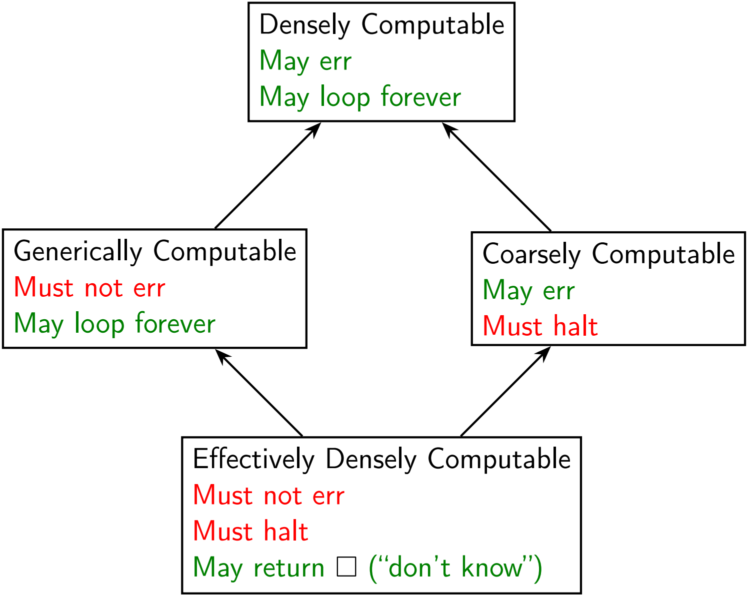
$$A = \{0^n \mid n \in \mathsf{HaltingProblem}\}$$

is not computable, but it is both coarsely and generically computable.

Example

Post's Correspondence Problem is not computable, but it is both coarsely and generically computable.

Four Horsemen of Asymptotic Computability



Coarse Reducibility

Definition

A is a **coarse approximation** of B if $A \triangle B$ is sparse.

Definition

A set A is **coarsely reducible** to a set B
(denoted $A \leq_{\mathrm{c}} B$)
if there's a Turing machine M such that,
for every coarse approximation C of B,
the set M^C is a coarse approximation of A.

Minimal Pairs

Definition

A pair of sets A and B form a **minimal pair** for Turing reducibility if neither A nor B are computable,
but if $C \leq_{\mathrm{T}} A$ and $C \leq_{\mathrm{T}} B$,
then C is computable.

Theorem (1950's)

There exists a minimal pair for the Turing degrees.

Minimal Pairs

Theorem (Hirschfeldt, Jockusch, Kuyper, Schupp, 2016)

There are measure-1 minimal pairs for coarse reducibility.

Theorem (Astor, Hirschfeldt, Jockusch, 2019)

There are measure-1 minimal pairs for dense reducibility.

Minimal Pairs

Theorem (Igusa, 2013)

There exists no minimal pairs for relative generic computability.

Theorem (Hirschfeldt, 2020)

There exists a minimal pair for generic reducibility.

Theorem (R)

There are only measure-0 many minimal pairs for generic reducibility.

Open Problem

Are there minimal pairs for effective dense reducibility?

Minimal Degrees

Definition

A sets A has a **minimal degree** for Turing reducibility if A is not computable, but if $C \leq_{\mathrm{T}} A$, then either $A \leq_{\mathrm{T}} C$ or C is computable.

Theorem (1960's)

There exists minimal Turing degrees.

Minimal Degrees

Theorem (R)

There exists a non-coarsely computable set A such that for every $C \leq_{\mathrm{c}} A$, either $C \geq_{\mathrm{c}} A$ or C can be coarsely approximated arbitrarily well by computable sets.

Open Problem

Are there minimal degrees for any of the asymptotic notions of computability?

Asymptotic Notions of Computability: Minimal Degrees and Minimal Pairs

Tiago Royer

University of Chicago

April 21st, 2024

https://cs.uchicago.edu/~royer/ams.pdf