The tree pigeonhole principle in the Weihrauch degrees

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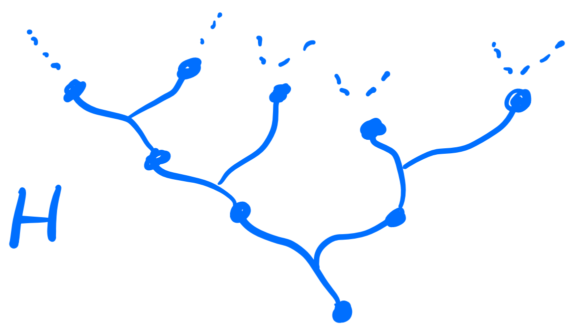
(with Damir Dzhafarov and Manlio Valenti)

Ramsey's theorem and the tree theorem

Ramsey's theorem for singletons. For every $k \geq 1$ and every coloring $c : \omega \to k$, there is an infinite set H such that $c \upharpoonright H$ is constant. We say H is a *homogeneous set* for c.

RT^1_k denotes this statement for a fixed k and RT^1 denotes $\forall k\, \mathsf{RT}^1_k$.

Tree pigeonhole principle. For every $k \geq 1$ and every coloring $c : 2^{<\omega} \to k$, there is an $H \subseteq 2^{<\omega}$ such that $H \cong 2^{<\omega}$ (as posets) and $c \upharpoonright H$ is constant.



TT^1_k denotes this statement for a fixed k and TT^1 denotes $\forall k\, \mathsf{TT}^1_k$.

TT^1_2: for every coloring $c : 2^{<\omega} \to 2$, there is an $H \cong 2^{<\omega}$ such that $c \restriction H$ is constant.

“Dense or cone” proof:

Case 1. Suppose the nodes with color 0 are dense in $2^{<\omega}$:

$$(\forall \sigma)(\exists \tau \succeq \sigma)\,(c(\tau) = 0)$$

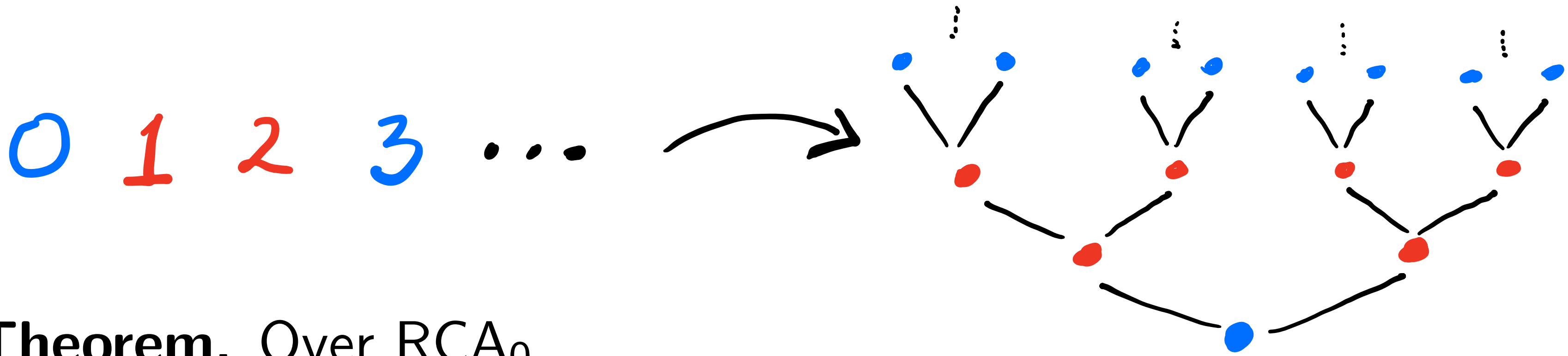
Define $h : 2^{<\omega} \to 2^{<\omega}$ recursively by $h(\sigma * i) = \tau$ such that $h(\sigma) * i \preceq \tau$ and $c(\tau) = 0$. Let $H = \text{range}(h)$.

Case 2. If the nodes with color 0 are not dense, then there is a σ such that $c(\tau) = 1$ for all $\tau \succeq \sigma$. Let $H = \{\tau \mid \sigma \preceq \tau\}$.

Fact. (Chubb, Hirst, McNicholl)

$$\mathsf{RCA}_0 \vdash (\forall k)(\mathsf{TT}^1_k \to \mathsf{RT}^1_k) \quad \text{and} \quad \mathsf{RCA}_0 \vdash \mathsf{TT}^1 \to \mathsf{RT}^1$$

Proof. Given $c : \omega \to k$, define $\widehat{c} : 2^{<\omega} \to k$ by $\widehat{c}(\sigma) = c(|\sigma|)$.



Theorem. Over RCA_0,

- Chubb, Hirst and McNicholl: $\mathsf{I}\Sigma_2 \to \mathsf{TT}^1 \to \mathsf{B}\Sigma_2$
- Corduan, Groszek and Mileti: $\mathsf{B}\Sigma_2 \not\to \mathsf{TT}^1$ (and more)
- Chong, Li, Wang and Yang: $\mathsf{TT}^1 \not\to \mathsf{I}\Sigma_2$ (and more)

Numerous questions about the first order consequences of TT^1 remain open. Our project considers these questions in the Weihrauch degrees.

Instance-solution problems

Many theorems studied in reverse math have the form

$$(\forall X)(\phi(X) \to (\exists \widehat{X})\psi(X, \widehat{X}))$$

where ϕ and ψ are arithmetical formulas (with set parameters).

Problem: given X such that $\phi(X)$, find $\widehat{X}$ such that $\psi(X, \widehat{X})$.

A *problem* is a partial multifunction $\mathsf{P} :\subseteq \omega^\omega \rightrightarrows \omega^\omega$.

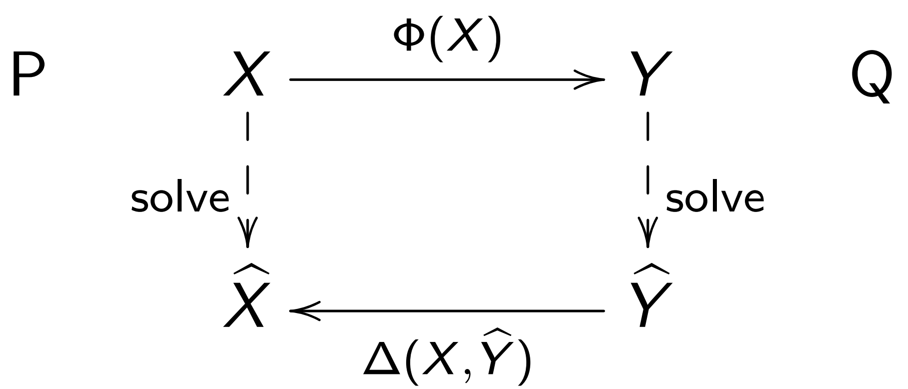
An $X \in \mathrm{dom}(\mathsf{P})$ is a P-*instance* and an $\widehat{X} \in \mathsf{P}(X)$ is a P-*solution* to X.

Example. An instance of TT^1_k is a coloring $c : 2^{<\omega} \to k$ and a solution to c is a homogeneous set $H \cong 2^{<\omega}$.

Weihrauch reducibility

For problems P and Q, P is *Weihrauch reducible* to Q if there are Turing functionals Φ and Δ such that

- for every P-instance X, $\Phi(X) = Y$ is a Q-instance, and
- for every Q-solution $\widehat{Y}$ to Y, $\Delta(X, \widehat{Y}) = \widehat{X}$ is a P-solution to X.



Example. For each k, $\mathrm{RT}^1_k \leq_W \mathrm{TT}^1_k$ by the level coloring translation.

The collection of all problems with $\leq_W$ form the Weihrauch degrees.

Translating RT^1 and TT^1 (Brattka and Rakotoniaina)

Perhaps the most natural translations of RT^1 and TT^1 are

- $\mathsf{RT}^1_{\mathbb{N}}$ instance is $c : \omega \to k$ for $k \geq 1$
  solution is an infinite homogeneous set for c.
- $\mathsf{TT}^1_{\mathbb{N}}$ instance is $c : 2^{<\omega} \to k$ for $k \geq 1$
  solution is an infinite homogeneous set $H \cong 2^{<\omega}$ for c.

Fact. $\mathsf{RT}^1_k \leq_W \mathsf{RT}^1_{\mathbb{N}}$ and $\mathsf{RT}^1_{\mathbb{N}} \leq_W \mathsf{TT}^1_{\mathbb{N}}$ by the level coloring of Chubb, Hirst and McNicholl.

(There is a second natural translation, RT^1_+ and TT^1_+, in which the instances are pairs $\langle k, c \rangle$ giving an explicit upper bound on the number of colors.)

Typically, the solution to a problem instance is a set or function.

- RT^1_k: instance is $c : \omega \to k$ and solution is a homogeneous set H.

However, some problems have numerical solutions.

- Infinite Pigeonhole Principle$_k$: instance is $c : \omega \to k$ and solution is a color $i < k$ such that $c^{-1}(i)$ is infinite.

Definition. P is a *first order problem* if every solution to a P-instance is a number (i.e. $\mathsf{P}(X) \subseteq \omega$ for all P-instances X).

All of the versions of Ramsey's theorem for singletons are Weihrauch equivalent to first order problems.

$$\mathsf{RT}^1_k \equiv_W \text{Inf Pigeon}_k \qquad \mathsf{RT}^1_{\mathbb{N}} \equiv_W \text{Inf Pigeon}_{\mathbb{N}}$$

First order part (Dzhafarov, Solomon, Yokoyama)

Proposition. For every P, there is a first order problem $^1\mathsf{P}$ s.t.

$$^1\mathsf{P} \equiv_W \sup_{\leq_W}\{\mathsf{F} \leq_W \mathsf{P} : \mathsf{F} \text{ is a first order problem}\} \leq_W \mathsf{P}$$

Example. $^1\mathsf{RT}^1_k \equiv_W \mathsf{RT}^1_k$ because Inf Pigeonhole$_k \equiv_W \mathsf{RT}^1_k$.

Similarly, $^1\mathsf{RT}^1_{\mathbb{N}} \equiv_W \mathsf{RT}^1_{\mathbb{N}}$

What can we say about $^1\mathsf{TT}^1_k$ and $^1\mathsf{TT}^1_{\mathbb{N}}$? Since $\mathsf{RT}^1_k \leq_W \mathsf{TT}^1_k$,

$$\mathsf{RT}^1_k \equiv_W {}^1\mathsf{RT}^1_k \leq_W {}^1\mathsf{TT}^1_k \leq_W \mathsf{TT}^1_k$$

$$\mathsf{RT}^1_{\mathbb{N}} \leq_W {}^1\mathsf{TT}^1_{\mathbb{N}} \leq_W \mathsf{TT}^1_{\mathbb{N}}$$

Is TT^1_k or $\mathsf{TT}^1_{\mathbb{N}}$ equivalent to a first order problem?

Exploring ${}^1\mathsf{TT}^1_2$

Recall the proof of TT^1_2: either $\{\tau : c(\tau) = 0\}$ is dense or there is a σ s.t. $c(\tau) = 1$ for all $\tau \succeq \sigma$.

V_0: instances $c : 2^{<\omega} \to 2$ and solutions are $\langle i, \sigma \rangle$ s.t. $\{\tau : c(\tau) = i\}$ is dense above σ.

V_4: instances $c : 2^{<\omega} \to 2$ and solutions are $\langle i, \sigma \rangle$ s.t. if $i = 0$, then $\{\tau : c(\tau) = 0\}$ is dense and if $i = 1$, then $c(\tau) = 1$ for all $\tau \succeq \sigma$.

We have $\mathsf{TT}^1_2 \leq_W V_0 \leq_W V_4$. However, . . .

Theorem (Dzhafarov, Solomon, Valenti). $V_0 <_W V_4$ and so $\mathsf{TT}^1_2 <_W V_4$.

In fact, $V_0 \equiv_W \mathsf{TC}_{\mathbb{N}} <_W V_4 \equiv_W \mathsf{sTC}_{\mathbb{N}}$.

Connections beyond $\mathsf{RT}^1_k \leq_W {}^1\mathsf{TT}^1_k \leq_W \mathsf{TT}^1_k$

Theorem (Dzhafarov, Solomon, Valenti). For every $k \geq 2$,

- $\mathsf{RT}^1_j \not\leq_W \mathsf{TT}^1_k$ for $j > k$. Therefore,

$$\mathsf{TT}^1_2 <_W \mathsf{TT}^1_3 <_W \mathsf{TT}^1_4 <_W \cdots$$

$${}^1\mathsf{TT}^1_2 <_W {}^1\mathsf{TT}^1_3 <_W {}^1\mathsf{TT}^1_4 <_W \cdots$$

- ${}^1\mathsf{TT}^1_k \not\leq_W \mathsf{RT}^1_j$ for every j, and therefore, $\mathsf{TT}^1_k \not\leq_W \mathsf{RT}^1_j$.
- $\mathsf{TT}^1_k \not\leq_W \mathsf{RT}^1_{\mathbb{N}}$, and therefore, $\mathsf{TT}^1_{\mathbb{N}} \not\leq_W \mathsf{RT}^1_{\mathbb{N}}$.
- However, $\mathsf{TT}^1_k \leq_W \mathsf{D}^2_k$.

Theorem (Dzhafarov, Solomon, Valenti). ${}^1\mathsf{TT}^1_{\mathbb{N}} \leq_W \mathsf{RT}^1_{\mathbb{N}}$

This result has a number of immediate corollaries.

- ${}^1\mathsf{TT}^1_{\mathbb{N}} \equiv_W \mathsf{RT}^1_{\mathbb{N}}$

  $$(\text{since } \mathsf{RT}^1_{\mathbb{N}} \leq_W {}^1\mathsf{TT}^1_{\mathbb{N}} \leq_W \mathsf{TT}^1_{\mathbb{N}})$$

- ${}^1\mathsf{TT}^1_{\mathbb{N}} <_W \mathsf{TT}^1_{\mathbb{N}}$, so $\mathsf{TT}^1_{\mathbb{N}}$ is not equivalent to a first order problem

  $$(\text{since } \mathsf{TT}^1_{\mathbb{N}} \not\leq_W \mathsf{RT}^1_{\mathbb{N}})$$

- For every $j, k \geq 2$, $\mathsf{TT}^1_k \not\leq_W {}^1\mathsf{TT}^1_j$

  $$(\text{since } {}^1\mathsf{TT}^1_j \leq_W {}^1\mathsf{TT}^1_{\mathbb{N}} \equiv_W \mathsf{RT}^1_{\mathbb{N}} \text{ but } \mathsf{TT}^1_k \not\leq_W \mathsf{RT}^1_{\mathbb{N}})$$

- ${}^1\mathsf{TT}^1_k <_W \mathsf{TT}^1_k$, so TT^1_k is not equivalent to a first order problem.

Thank you!