Universal Sets for Projections

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Geometric Measure Theory

- Size of small/irregular **sets**
- Fractal dimensions - Hausdorff, packing, etc.

Algorithmic Randomness

- Inherent randomness of **points** (binary sequences)
- Computability theory

We can use computability theory to solve problems in **classical** geometric measure theory

Effective Dimensions of Points

Let $x \in \mathbb{R}$ and $r \in \mathbb{N}$. The *Kolmogorov complexity of x at precision r* is

$$\begin{aligned} K_r(x) &\approx \text{ Kolmogorov complexity of the first r bits of binary expansion of } x \\ &= \text{ length of shortest program outputting the first r bits} \\ &\qquad\qquad\text{of the binary expansion of } x. \end{aligned}$$

- Can generalize this to $\mathbb{R}^n$ in the natural way.
- Can relativize this to arbitrary oracles.

Effective Dimensions of Points

Let $n \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The *(effective Hausdorff) dimension of* x is

$$\dim(x) = \liminf_{r \to \infty} \frac{K_r(x)}{r}.$$

Let $n \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The *effective packing dimension of* x is

$$\mathrm{Dim}(x) = \limsup_{r \to \infty} \frac{K_r(x)}{r}.$$

The Point-to-Set Principle

Theorem (J. Lutz and N. Lutz, '16)

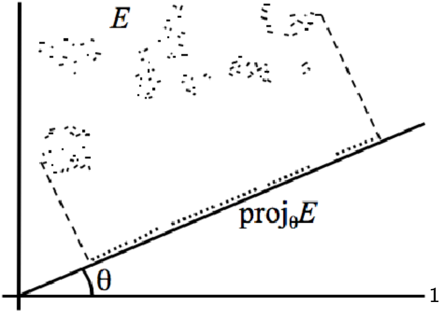
For every set $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x), \text{ and}$$

$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \mathrm{Dim}^A(x).$$

- The Hausdorff and packing dimension of a *set* is characterized by the corresponding dimension of the *points* in the set.

Orthogonal Projections



If E is big, is it true that the projection of E is big?

[1] Kenneth Falconer, Sixty years of fractal projections

Marstrands Projection Theorem

In the plane, we parameterize orthogonal projections with the angle the line makes with the x-axis, so

$$p_\theta : \mathbb{R}^2 \to \mathbb{R}$$
$$p_\theta(x, y) = x \cos\theta + y \sin\theta.$$

Marstrands projection theorem

Theorem (Marstrand '54)

Let $E \subseteq \mathbb{R}^2$ be an analytic set with $\dim_H(E) = s$. Then for almost every $\theta \in (0, 2\pi)$,

$$\dim_H(p_\theta E) = \min\{s, 1\}.$$

- Any subset of a line has Hausdorff dimension at most 1.
- Lipschitz functions (like p_θ) cannot increase the Hausdorff dimension of a set.
  - MPT shows that for a.e. angle, $\dim_H(p_\theta E)$ is maximal.
- Useful in many different problems in geometric measure theory
- (S. '21) Can weaken the assumption that E is analytic to the assumption that E has *optimal oracles*.

Universal sets

Theorem (Marstrand '54)

Let $E \subseteq \mathbb{R}^2$ be an analytic set with $\dim_H(E) = s$. *Then for almost every* $\theta \in (0, 2\pi)$,

$$\dim_H(p_\theta E) = \min\{s, 1\} .$$

Definition: Let $\mathcal{C}$ be a class of subsets of $\mathbb{R}^2$. We say a subset $S \subseteq [0, 2\pi)$ is *universal for* $\mathcal{C}$ if, for every $E \in \mathcal{C}$, there is an angle $\theta \in S$ such that

$$\dim_H(p_\theta E) = \min\{s, 1\}.$$

Question: Are there small universal sets?

Universal sets

Theorem (Fiedler, S.)

There is a $\mathbf{\Sigma}^1_1$-universal set S such that $\mu(S) = 0$.

- We actually prove that the set

$$S = \{\theta \in [0, 2\pi) \mid \theta \text{ is not ML random } \}.$$

is $\mathbf{\Sigma}^1_1$-universal.

- We prove something slightly stronger: For every analytic set E, "most" directions in S satisfy Marstrand's theorem.

- We can weaken the class slightly - the same theorem holds for the class $\mathcal{C}_{OO}$ of sets in $\mathbb{R}^2$ with *optimal oracles*.

Universal sets

Generalize this in two ways:

- Assuming more *regularity* we can construct smaller universal sets:

Theorem (Fiedler, S.)

Let $\mathcal{A}$ be the class of Ahlfors-David regular sets of $\mathbb{R}^2$. There is a $\mathcal{A}$-universal set S such that $\dim_H(S) = 0$.

- Assuming the sets have lower dimension, we can construct smaller universal sets:

Theorem (Fiedler, S.)

Let $\mathcal{C}_s$ be the class of analytic sets with Hausdorff dimension at most $s < 1$. There is a $\mathcal{C}_s$-universal set S such that $\dim_H(S) = s$.

The End

Thank you!