On the existence of universal numberings for families of d.c.e. sets

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CONTENTS

Computable Numberings and Reducibilities of Numberings

- A mapping $\alpha : \omega \to \mathcal{A}$ of the set ω of natural numbers onto a family $\mathcal{A}$ of c.e. sets is called a *computable numbering* of $\mathcal{A}$ if the set $\{\langle x, n\rangle \mid x \in \alpha(n)\}$ is c.e. And a family $\mathcal{A}$ of subsets of ω is called *computable* if it has a computable numbering.

  A computable family $\mathcal{A}$ is a *uniformly c.e. class* of sets, and every computable numbering α of $\mathcal{A}$ defines a uniform c.e. sequence $\alpha(0), \alpha(1), \ldots$ of the members of $\mathcal{A}$ (possibly with repetition).

- A numbering α is called *reducible* to a numbering β (in symbols, $\alpha \leqslant \beta$) if $\alpha = \beta \circ f$ for some computable function f. Two numberings α, β are called *equivalent* if they are reducible to each other.

About Universal (Principal) Numbering

- The notion Com($\mathcal{A}$) stands for all computable numberings of a computable family $\mathcal{A}$ of c.e. sets.
- A universal (principal) numbering for a class of numberings is a numbering in the class which can simulate any numbering in the class.
- More precisely, a numbering $\alpha : \omega \to \mathcal{A}$ is called *universal (principal)* if $\alpha \in \mathrm{Com}(\mathcal{A})$ and $\beta \leqslant \alpha$ for each numbering $\beta \in \mathrm{Com}(\mathcal{A})$.

  There is exist interesting sufficient condition for a subset $S \subseteq \mathcal{A}$ to be universal in $(\mathcal{A}, \alpha)$.
- $S \subseteq \mathcal{A}$ is called *wn-subset* of $(\mathcal{A}, \alpha)$, if there is exists a partial computable function f such that $dom(f)_{\supseteq}\alpha^{-1}(S), \alpha f(n) \in S$ for all $n \in dom(f)$, and if $n \in \alpha^{-1}(S)$, then $\alpha(n) = \alpha f(n)$.

Examples of Principal Numberings

- If we consider the computable numberings of the unary partial computable functions, i.e. the uniformly computable sequences $\psi_0, \psi_1, \ldots$ of the unary partial computable functions, then the standard Gödel numbering $\varphi_0, \varphi_1, \ldots$ is a classical example of a principal numbering, since for any such sequence, $\psi_e = \varphi_{f(e)}$ for some computable function f and all $e \in \omega$.
- Analogously, the standard Gödel numbering $\{W_e\}_{e \in \omega}$ of the c.e. sets is another example of a principal numbering for the class of c.e. sets.

Ways of Constructing Principal Numberings

For a given computable family $\mathcal{A}$ of c.e. sets, two main ways of constructing principal numberings are known.

- The first way is based on the idea of considering uniform computations of all computable numberings, or at least of witnesses from each equivalence class of numberings, lying in $\mathrm{Com}(\mathcal{A})$. Essentially, this way is epitomized in Rice's description of the classes of c.e. sets whose index sets in W are c.e.
- The second way originated from the notion of a *standard class*, introduced by A.Lachlan. Generalizations of the notion of standard class by A.I.Mal'tsev and Yu.L. Ershov provided very fruitful tools for constructing principal numberings.

Now we formulate one of the finest results on principal numberings.

wn-subset

Theorem (Lachlan)

Every finite family of c.e. sets has a universal numbering.

- A family $S \subseteq \mathcal{A}$ has a universal computable numbering iff S is a universal subset of $(\mathcal{A}, \alpha)$.
- *A finite family $S \subseteq \mathcal{A}$ is wn-subset of $(\mathcal{A}, \alpha)$ and hence is universal subset of $(\mathcal{A}, \alpha)$.*

Computable Numberings in Hierarchies

The notion of *d.c.e.* and *n.c.e.* sets goes back to Putnam [1965] and Gold [1965] and was first investigated and generalized by Ershov [1968a,b, 1970]. The arising hierarchy of sets is now known as the Ershov difference hierarchy.

S.S. Goncharov and A.Sorbi offered a general approach for studying classes of objects which admit a constructive description in a formal language via a Gödel numbering for formulas of the language. According to their approach, a numbering is computable if there exists a computable function which, for every object and each index of this object in the numbering, produces some Gödel index of its constructive description.

Computable Numberings in Hierarchies

- Σ_n^{-1} is the class of level n of the Ershov hierarchy of sets (n-c.e. sets).
- Σ_n^0 is the class of level n of the arithmetical hierarchy.

The notion of a computable numbering for a family $\mathcal{A}$ of sets in the class Σ_n^i, with $i \in \{$-1,0$\}$, may be deduced from the Goncharov–Sorbi approach as follows.

- A numbering α of a family $\mathcal{A} \subseteq \Sigma_n^i$ is *Σ_n^i-computable* if $\{\langle x, m\rangle : x \in \alpha(m)\} \in \Sigma_n^i$, i.e. the sequence $\alpha(0), \alpha(1), \ldots$ of the members of $\mathcal{A}$ is uniformly Σ_n^i.
- The set of all Σ_n^i-computable numberings of a family $\mathcal{A} \subseteq \Sigma_n^i$ denote by $\text{Com}_n^i(\mathcal{A})$.

Universal numberings

Since $\mathcal{A} \subseteq \Sigma^i_n$ implies $\mathcal{A} \subseteq \Sigma^i_m$ for all $m > n$, it follows that we should be careful in defining the notion of principal numbering.

Definition

Let $\mathcal{A} \subseteq \Sigma^i_n$ *and let* $m \geq n$*. A numbering* $\alpha : \omega \to \mathcal{A}$ *is called universal in* $\mathrm{Com}^i_m(\mathcal{A})$ *if* $\alpha \in \mathrm{Com}^i_m(\mathcal{A})$ *and* $\beta \leqslant \alpha$ *for all* $\beta \in \mathrm{Com}^i_m(\mathcal{A})$.

Computing the Sets $\alpha(e)$

Let $A(n, x, t)$ denote a function satisfying the following conditions:

- $\mathrm{range}(A) \subseteq \{0, 1\}$;
- $A(e, x, 0) = 0$, for all e and x.

We can treat this function as uniform procedure for computing the sets $\alpha(e)$. Given e and x, $A(e, x, 0) = 0$ means that initially the number x is not enumerated into $\alpha(e)$. The number x stays outside of $\alpha(e)$ until the function $\lambda t A(e, x, t)$ changes its value from 0 to 1. When this happens, the number x is enumerated into $\alpha(e)$. Now, x remains in $\alpha(e)$ until $\lambda t A(e, x, t)$ changes the value from 1 to 0. In this case, the number x is taken off the set $\alpha(e)$. And again we wait for the value of $\lambda t A(e, x, t)$ to change from 0 to 1, to put x into $\alpha(e)$ for the second time, and so on.

Some Criteria

- For $\mathcal{A} \subseteq \Sigma^0_1$, a numbering α is Σ^0_1-computable if and only if there exists a computable function A such that, for all e, x, $\lambda t A(e, x, t)$ is a function **monotonic** in t, and

$$x \in \alpha(e) \iff \lim_t A(e, x, t) = 1.$$

- If $\mathcal{A} \subseteq \Sigma^{-1}_{n+1}$ then a numbering α is Σ^{-1}_{n+1}-computable if and only if there exists a computable function A such that, for all e, x,

$$|\{t : A(e, x, t+1) \neq A(e, x, t)\}| \leq n+1$$

.

- For a Σ^i_n-computable numbering α, we say that such a computable function A *represents* a Σ^i_n computation of $\alpha(e)$.

Theorem 2

Note that the computable function $A(e, x, t)$ above is monotonic in t only in the classical case of c.e. sets (i.e. $\mathcal{A} \subseteq \Sigma^0_1$). It seems that the non-monotonic behavior of this function is the main reason for Theorem 1 to fail in all non-classical cases. We recall the following known result.

Theorem (Badaev, Goncharov, Sorbi, [2003])

Let $\mathcal{A}$ be any finite family of Σ^0_{n+2} sets. Then $\mathcal{A}$ has an universal numbering in $\mathrm{Com}^i_{n+2}(\mathcal{A})$ if and only if $\mathcal{A}$ contains a least set under inclusion.

Universal Numberings for Finite Families of the n.c.e. sets.

Theorem 3

Theorem

For every n, the class Σ^{-1}_{n+2} of the Ershov hierarchy has a universal numbering in $\mathrm{Com}^{-1}_{n+2}(\Sigma^{-1}_{n+2})$.

We will denote this universal numbering by $W^{(-1,n+2)}$.

wn-subset

Definition

A family $\mathcal{A} \subseteq \Sigma_k^i$ is called a wn-subset of Σ_k^{-1} if there exist a c.e. set I and a sequence $\{V_e\}_{e\in\omega}$ such that

1. *I contains the index set of the family $\mathcal{A}$ with respect to the numbering $W^{(-1,k)}$;*
2. *V is a Σ_k^{-1}-computable numbering;*
3. *for every $e \in I$, $V_e \in \mathcal{A}$;*
4. *for every $e \in I$, if $W_e^{(-1,k)} \in \mathcal{A}$ then $V_e = W_e^{(-1,k)}$.*

Lemma

If a family $\mathcal{A} \subseteq \Sigma_k^i$ is a wn-subset of Σ_k^{-1} then $\mathcal{A}$ has a universal numbering in $\mathrm{Com}_k^{-1}(\mathcal{A})$.

Previous result.

Theorem (Abeshev, Badaev [2009])

Let $k > 1$ and $m > 0$ be any numbers. If $F_0, F_1, \ldots, F_m$ is a sequence of finite sets and $B \in \Sigma_k^{-1}$ is a set such that no F_i in the sequence intersects B, then the family $\mathcal{A} = \{B \cup F_i : i \leq m\}$ is a *wn*-subset of Σ_k^{-1}.

Questions:

1. Do there exist finite families of *n.c.e.* sets (Ershov hierarchy) without universal numberings?

2. Do there exist other finite families of *n.c.e.* sets (Ershov hierarchy) with universal numberings?

3. What is the criteria of finding universal numberings of finite families of *n.c.e.* sets?

Results.

Theorem (A.)

There is a family $\mathcal{F} = \{A, B\}$ of nonempty, disjoint d.c.e. sets such that the family $\mathcal{F}$ has no universal numbering.

Theorem (A.)

If there are c.e. sets A_0, A_1, B_0, B_1 and $A = A_0 \setminus A_1$ and $B = B_0 \setminus B_1$ such that

$$\forall x\,(x \in A_0 \Rightarrow x \notin A_1 \text{ or } x \notin B),$$
$$\forall x\,(x \in B_0 \Rightarrow x \notin B_1 \text{ or } x \notin A),$$

then there is a universal numbering π for $\mathcal{F} = \{A, B\}$.

Results.

Theorem (*)(A.)

If A and B are d.c.e. sets with $A \not\subseteq B$ and $B \not\subseteq A$ and $A_0 \supseteq A$ and $B_0 \supseteq B$ are c.e. sets with $A_0 \cap B = A \cap B = A \cap B_0$ then there is a universal numbering π for $\mathcal{F} = \{A, B\}$.

Theorem (A.)

The condition of **(Theorem (*))** is not necessary. There are d.c.e. sets A and B with a universal numbering π of $\{A, B\}$ with $A \not\subseteq B$ and $B \not\subseteq A$ such that for all c.e. sets $A_0 \supseteq A$ we have $A_0 \cap B \neq A \cap B$ and for all c.e. sets $B_0 \supseteq B$ we have $A \cap B_0 \neq A \cap B$.

Results.

Theorem (A.)

If there is an enumeration of the family $\{A, B\}$ of d.c.e. sets with $A \not\subseteq B$ and $B \not\subseteq A$ such that the sets

$$\widetilde{A} = \{x | \exists s_0 < s_1 < s_2 (x \in B_{s_0} \ \& \ x \notin A_{s_1} \cup B_{s_1} \ \& \ x \in A_{s_2})\},$$
$$\widetilde{B} = \{x | \exists s_0 < s_1 < s_2 (x \in A_{s_0} \ \& \ x \notin A_{s_1} \cup B_{s_1} \ \& \ x \in B_{s_2})\},$$

are computable then there is a universal numbering π for $\mathcal{F} = \{A, B\}$.

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THANK YOU FOR YOUR ATTENTION!